If V^* is a Lyapunov Function (LF), V^*'s infinitesimal is the Lyapunov Equation (LE), i.e., $\min \dot{V}^* = 0$

Take as example
$$\begin{cases} V_i(x_i(t)) = \int_t^\infty r_i(x_i(\tau), u_i(\tau))d\tau \\ r_i(x_i(t), u_i(t)) = Q_i(x_i) + u_i^T(x_i) R_i u_i(x_i) \\ \dot{x} = f_i(x_i(t)) + g_i(x_i(t)) u_i(x_i(t)) \end{cases}$$

If $V_i(x_i(t))$ is a LF, $\underline{0 = \min \dot{V}_i(x_i(t))}$ is the LE.

$$\to \begin{cases} \frac{\partial V_i}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial V_i}{\partial x}\frac{\partial x}{\partial t} = 0 \Rightarrow r_i(x_i(t), u_i(t)) + (\nabla V_i(x_i))^T \dot{x} = 0 \\ \nabla V(t, x) = 0 \end{cases}$$

$$\Downarrow$$

$$r_i(x_i, u_i) + (\nabla V_i(x_i))^T [f_i(x_i) + g_i(x_i) u_i(x_i)] = 0$$

HJB eq. is $\min_{u \in \Psi} H_i(x_i, u_i, \nabla^* V_i(x_i)) = 0$

Hamiltonian Function (HF) is $H_i(x_i, u_i, \nabla V_i) = r_i(x_i, u_i) + (\nabla V_i(x_i))^T [f_i(x_i) + g_i(x_i) u_i(x_i)]$

Solve $u_i^*(x_i) = \arg\min_{u \in \Psi} H(x_i, u, \nabla V) = -\frac{1}{2} R_i^{-1} g_i^T(x_i) \nabla V_i^*(x_i)$

$$\Downarrow$$

HJB eq in terms of $\nabla V_i^*(x_i)$ $\longleftarrow$ $0 = Q_i(x_i) + (\nabla V_i^*(x_i))^T f_i(x_i) - \frac{1}{4} (\nabla V_i^*(x_i))^T g_i(x_i) R_i^{-1} g_i^T(x_i) \nabla V_i^*(x_i)$

Theorem 1. Consider $\Sigma_i : \dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(x_i(t))$, subsystem (isolated)

$\forall \pi_i \geq \frac{1}{2}$, feedback control ensure that closed-loop isolated subsystem asymptotically stable.

$$u_i(x_i) = \pi_i u_i^*(x_i) = -\frac{1}{2}\pi_i R_i^{-1} g^T(x_i)\nabla V_i^*(x_i)$$

Proof.

If we can prove $V_i^*(x_i)$ is a Lyapunov function, $\begin{cases} V_i^* > 0 \\ \dot{V}_i^* < 0 \end{cases}$, Lyapunov local stable.

$$\begin{aligned} \dot{V}_i^*(x_i) = (\nabla V_i^*(x_i))^T \dot{x}_i &= (\nabla V_i^*(x_i))^T [f_i(x_i) + g_i(x_i)u_i(x_i)] \\ &+ \frac{1}{2}(\nabla V_i^*(x_i))^T g_i(x_i) u_i^*(x_i) \\ &- \frac{1}{2}(\nabla V_i^*(x_i))^T g_i(x_i) u_i^*(x_i) \end{aligned}$$

$$-\frac{1}{4}(\nabla V_i^*(x_i))^T g_i(x_i) R_i^{-1} g_i^T(x_i)\nabla V_i^*(x_i) = \frac{1}{2}(\nabla V_i^*(x_i))^T g_i(x_i) \; \underline{-\frac{1}{2}R_i^{-1} g_i^T(x_i)\nabla V_i^*(x_i)} \longleftrightarrow u_i^*(x_i)$$

$$= (\nabla V_i^*(x_i))^T f_i(x_i) + (\nabla V_i^*(x_i))^T g_i(x_i) u_i^*(x_i)(\pi_i - \tfrac{1}{2}) + \tfrac{1}{2}(\nabla V_i^*(x_i))^T g_i(x_i) u_i^*(x_i)$$

$$= (\nabla V_i^*(x_i))^T f_i(x_i) + (\nabla V_i^*(x_i))^T g_i(x_i)[-\tfrac{1}{2}R_i^{-1} g_i^T(x_i)\nabla V_i^*(x_i)](\pi_i - \tfrac{1}{2}) + \tfrac{1}{2}(\nabla V_i^*(x_i))^T g_i(x_i)[-\tfrac{1}{2}R_i^{-1} g_i^T(x_i)\nabla V_i^*(x_i)].$$

$$= (\nabla V_i^*(x_i))^T f_i(x_i) - \tfrac{1}{2}(\pi_i - \tfrac{1}{2})(\nabla V_i^*(x_i))^T g_i(x_i) R^{-1} g_i^T(x_i)\nabla V_i^*(x_i) - \tfrac{1}{4}(\nabla V_i^*(x_i))^T g_i(x_i) R_i^{-1} g_i^T(x_i)\nabla V_i^*(x_i)$$

$\because$ HJB :

$\hookrightarrow$ $0 = Q_i(x_i) + (\nabla V_i^*(x_i))^T f_i(x_i) - \frac{1}{4}(\nabla V_i^*(x_i))^T g_i(x_i) R_i^{-1} g_i^T(x_i)\nabla V_i^*(x_i)$

$\therefore$
$$\dot{V}_i^*(x_i) = (\nabla V_i^*(x_i))^T f_i(x_i) - \tfrac{1}{2}(\pi_i - \tfrac{1}{2})(\nabla V_i^*(x_i))^T \| R_i^{-\frac{1}{2}} g^T(x_i)\nabla V_i^*(x_i)\|^2 - [Q_i(x_i) + (\nabla V_i^*(x_i))^T f_i(x_i)]$$

$$= -Q_i(x_i) - \tfrac{1}{2}(\pi_i - \tfrac{1}{2})\| R_i^{-\frac{1}{2}} g_i^T(x_i)\nabla V_i^*(x_i)\|^2 \qquad (11)$$

$\therefore$ $\dot{V}_i^*(x_i) < 0 \quad \forall \pi_i \geq \frac{1}{2}$

$\because$ $V_i^*(x_i) > 0$

$\hookrightarrow$ Lyapunov function

$\Big\}$ $\Rightarrow$ Lyapunov local stability theory conditions ✓

(Q.E.D.)

Theorem 2. $\pi_i > \pi_i^*$, feedback controls $u_i(x_i) = \pi_i u_i^*(x_i) = -\frac{1}{2}\pi_i R_i^{-1} g_i^T(x_i) \nabla V_i^*(x_i)$ ensure closed-loop interconnected system asymptotically stable.

$[u_1(x_1), u_2(x_2), \cdots, u_N(x_N)]$ is decentralized control law of the interconnected system.

According to Theorem 1, $V_i^*(x_i)$ is Lyapunov function.

$$\Rightarrow \quad L(x) = \sum_{i=1}^{N} \theta_i V_i^*(x_i)$$

$$\dot{L}(x) = \sum_{i=1}^{N} \theta_i \dot{V}_i^*(x_i) = \sum_{i=1}^{N} \theta_i \frac{\partial V_i^*(x_i)}{\partial t} = \sum_{i=1}^{N} \theta_i \frac{\partial V_i^*(x_i)}{\partial x}\frac{\partial x}{\partial t} = \sum_{i=1}^{N} \theta_i (\nabla V_i^*(x_i))^T \dot{x} = \sum_{i=1}^{N} \theta_i (\nabla V_i^*(x_i))^T [f_i(x_i) + g_i(x_i)[u_i(x_i) + z_i(x_i)]]$$

$$= \sum_{i=1}^{N} \theta_i \{ \underbrace{(\nabla V_i^*(x_i))^T [f_i(x_i) + g_i(x_i) u_i(x_i)}_{\dot{V}_i^*(x_i)} + g_i(x_i) z_i(x)] \}$$

$$= \sum_{i=1}^{N} \theta_i [\dot{V}_i^*(x_i) + (\nabla V_i^*(x_i))^T g_i(x_i) z_i(x)]$$

$$\overset{(11)}{=} \sum_{i=1}^{N} \theta_i [-Q_i(x_i) - \frac{1}{2}(\pi_i - \frac{1}{2}) \| R_i^{-\frac{1}{2}} g_i^T(x_i) \nabla V_i^*(x_i) \|^2 + (\nabla V_i^*(x_i))^T g_i(x_i) z_i(x)]$$

$$= \sum_{i=1}^{N} \theta_i [-Q_i(x_i) + \frac{1}{2}(\pi_i - \frac{1}{2}) \| R_i^{-\frac{1}{2}} g_i^T(x_i) \nabla V_i^*(x_i) \|^2 + (\nabla V_i^*(x_i))^T g_i(x_i) R_i^{\frac{1}{2}} \bar{z}_i(x)]$$

$$= -\sum_{i=1}^{N} \theta_i [Q_i(x_i) + \frac{1}{2}(\pi_i - \frac{1}{2}) \| R_i^{-\frac{1}{2}} g_i^T(x_i) \nabla V_i^*(x_i) \|^2 - (\nabla V_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}} \bar{z}_i(x)]$$

$V = J$

$$\leq -\sum_{i=1}^{N} \theta_i [Q_i(x_i) + \frac{1}{2}(\pi_i - \frac{1}{2}) \| (\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}} \|^2 - \| \nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}} \| \sum_{j=1}^{N} \lambda_{ij} Q^{\frac{1}{2}}(x_j)] \qquad (17)$$

$$
\begin{array}{ccc}
\xi^T & A & \xi
\end{array}
$$

$$
\left[Q_1(x_1)^{\frac{1}{2}} \; Q_2(x_2)^{\frac{1}{2}} \; \dots \; \|\nabla J_N^*(x_N)^T g_N(x_N) R_N^{-\frac{1}{2}}\| \right] \times
\begin{bmatrix}
\theta_1 & & & & -\frac{1}{2}\theta_1\lambda_{11} & -\frac{1}{2}\theta_2\lambda_{21} & \cdots & -\frac{1}{2}\theta_N\lambda_{N1} \\
& \theta_2 & & & -\frac{1}{2}\theta_1\lambda_{12} & -\frac{1}{2}\theta_2\lambda_{22} & \cdots & -\frac{1}{2}\theta_N\lambda_{N2} \\
& & \ddots & & \vdots & \vdots & \ddots & \vdots \\
& & & \theta_N & -\frac{1}{2}\theta_1\lambda_{1N} & -\frac{1}{2}\theta_2\lambda_{2N} & \cdots & -\frac{1}{2}\theta_N\lambda_{NN} \\
-\frac{1}{2}\theta_1\lambda_{11} & -\frac{1}{2}\theta_1\lambda_{12} & \cdots & -\frac{1}{2}\theta_1\lambda_{1N} & \frac{1}{2}\theta_1(\pi_1-\frac{1}{2}) & & & \\
-\frac{1}{2}\theta_2\lambda_{21} & -\frac{1}{2}\theta_2\lambda_{22} & \cdots & -\frac{1}{2}\theta_2\lambda_{2N} & & \frac{1}{2}\theta_2(\pi_2-\frac{1}{2}) & & \\
\vdots & \vdots & \ddots & \vdots & & & \ddots & \\
-\frac{1}{2}\theta_N\lambda_{N1} & -\frac{1}{2}\theta_N\lambda_{N2} & \cdots & -\frac{1}{2}\theta_N\lambda_{NN} & & & & \frac{1}{2}\theta_N(\pi_N-\frac{1}{2})
\end{bmatrix}
\times
\begin{bmatrix}
Q_1(x_1)^{\frac{1}{2}} \\
Q_2(x_2)^{\frac{1}{2}} \\
\vdots \\
Q_N(x_N)^{\frac{1}{2}} \\
\|(\nabla J_1(x_1))^T g_1(x_1) R_1^{-\frac{1}{2}}\| \\
\|(\nabla J_2(x_2))^T g_2(x_2) R_2^{-\frac{1}{2}}\| \\
\vdots \\
\|(\nabla J_N(x_N))^T g_N(x_N) R_N^{-\frac{1}{2}}\|
\end{bmatrix}
$$

$\boxed{1 \times 2N}$

$$
= \Big[\theta_1 Q_1(x_1)^{\frac{1}{2}} + \sum_{i=1}^{N} \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| (-\tfrac{1}{2}\theta_i\lambda_{i1})
\quad \theta_2 Q_2(x_2)^{\frac{1}{2}} + \sum_{i=1}^{N} \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| (-\tfrac{1}{2}\theta_i\lambda_{i2}) \quad \cdots
$$

$$
\cdots \; \theta_N Q_N(x_N)^{\frac{1}{2}} + \sum_{i=1}^{N} \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| (-\tfrac{1}{2}\theta_i\lambda_{iN})
$$

$$
\sum_{i=1}^{N} (-\tfrac{1}{2}\theta_1)(Q_i(x_i)^{\frac{1}{2}}\lambda_{1i}) + \tfrac{1}{2}\theta_1(\pi_1-\tfrac{1}{2})\|(\nabla J_1^*(x_1))^T g_1(x_1) R_1^{-\frac{1}{2}}\|
$$

$$
\sum_{i=1}^{N} (-\tfrac{1}{2}\theta_2)(Q_i(x_i)^{\frac{1}{2}}\lambda_{2i}) + \tfrac{1}{2}\theta_2(\pi_2-\tfrac{1}{2})\|(\nabla J_2^*(x_2))^T g_2(x_2) R_2^{-\frac{1}{2}}\| \quad \cdots
$$

$$
\cdots \; \sum_{i=1}^{N} (-\tfrac{1}{2}\theta_N)(Q_i(x_i)^{\frac{1}{2}}\lambda_{Ni}) + \tfrac{1}{2}\theta_N(\pi_N-\tfrac{1}{2})\|(\nabla J_N^*(x_N))^T g_N(x_N) R_N^{-\frac{1}{2}}\| \Big] \times \xi
$$

$$
\begin{aligned}
= {} & \theta_1 Q_1(x_1)^{\frac{1}{2}} Q_1(x_1)^{\frac{1}{2}} + \sum_{i=1}^{N} \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| (-\tfrac{1}{2}\theta_i\lambda_{i1}) Q_1(x_1)^{\frac{1}{2}} \\
& + \theta_2 Q_2(x_2)^{\frac{1}{2}} Q_2(x_2)^{\frac{1}{2}} + \sum_{i=1}^{N} \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| (-\tfrac{1}{2}\theta_i\lambda_{i2}) Q_2(x_2)^{\frac{1}{2}} + \cdots \\
& \cdots + \theta_N Q_N(x_N)^{\frac{1}{2}} Q_N(x_N)^{\frac{1}{2}} + \sum_{i=1}^{N} \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| (-\tfrac{1}{2}\theta_i\lambda_{iN}) Q_N(x_N)^{\frac{1}{2}} \\
& + \sum_{i=1}^{N} (-\tfrac{1}{2}\theta_1)(Q_i(x_i)^{\frac{1}{2}}\lambda_{1i})\|(\nabla J_1(x_1))^T g_1(x_1) R_1^{-\frac{1}{2}}\| + \tfrac{1}{2}\theta_1(\pi_1-\tfrac{1}{2})\|(\nabla J_1^*(x_1))^T g_1(x_1) R_1^{-\frac{1}{2}}\|^2 \\
& \sum_{i=1}^{N} (-\tfrac{1}{2}\theta_2)(Q_i(x_i)^{\frac{1}{2}}\lambda_{2i})\|(\nabla J_2(x_2))^T g_2(x_2) R_2^{-\frac{1}{2}}\| + \tfrac{1}{2}\theta_2(\pi_2-\tfrac{1}{2})\|(\nabla J_2^*(x_2))^T g_2(x_2) R_2^{-\frac{1}{2}}\|^2 + \cdots \\
& \cdots + \sum_{i=1}^{N} (-\tfrac{1}{2}\theta_N)(Q_i(x_i)^{\frac{1}{2}}\lambda_{Ni})\|(\nabla J_N(x_N))^T g_N(x_N) R_N^{-\frac{1}{2}}\| + \tfrac{1}{2}\theta_N(\pi_N-\tfrac{1}{2})\|(\nabla J_N^*(x_2))^T g_N(x_N) R_N^{-\frac{1}{2}}\|^2
\end{aligned}
$$

$$-\xi^T A \xi = -\xi^T \begin{bmatrix} \Theta & -\frac{1}{2}\Lambda^T \Theta \\ -\frac{1}{2}\Theta\Lambda & \Theta\Pi \end{bmatrix} \xi$$

$$= \sum_{i=1}^{N} \theta_i Q_i(x_i) - \sum_{i=1}^{N} \theta_i \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \frac{1}{2}\Big[\sum_{j=1}^{N} Q_j(x_j)^{\frac{1}{2}} \lambda_{ij}\Big] + \sum_{i=1}^{N} \left(-\frac{1}{2}\right) \theta_i \|(\nabla J_i(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \sum_{j=1}^{N} \lambda_{ji} Q_j(x_i)^{\frac{1}{2}}$$

$$+ \sum_{i=1}^{N} \frac{1}{2} \theta_i \left(\pi_i - \frac{1}{2}\right) \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\|^2$$

$$= \sum_{i=1}^{N} \theta_i \Big[Q_i(x_i) - \frac{1}{2}\|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \Big[\sum_{j=1}^{N} Q_j(x_j)^{\frac{1}{2}} \lambda_{ij}\Big] - \frac{1}{2}\|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \Big[\sum_{j=1}^{N} Q_j(x_j)^{\frac{1}{2}} \lambda_{ji}\Big]$$

$$+ \frac{1}{2}\left(\pi_i - \frac{1}{2}\right) \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\|^2 \Big]$$

$$= \sum_{i=1}^{N} \theta_i \Big[Q_i(x_i) - \frac{1}{2}\|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \sum_{j=1}^{N} Q_j(x_j)^{\frac{1}{2}} (\lambda_{ij} + \lambda_{ji}) + \frac{1}{2}\left(\pi_i - \frac{1}{2}\right)\|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\|^2 \Big]$$

$$= \sum_{i=1}^{N} \theta_i \Big[Q_i(x_i) + \frac{1}{2}\left(\pi_i - \frac{1}{2}\right)\|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\|^2 - \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \sum_{j=1}^{N} Q_j(x_j)^{\frac{1}{2}} \frac{1}{2}(\lambda_{ij} + \lambda_{ji}) \Big]$$

$$= \sum_{i=1}^{N} \theta_i \Big[Q_i(x_i) + \frac{1}{2}\left(\pi_i - \frac{1}{2}\right)\|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\|^2 - \|(\nabla J_i^*(x_i))^T g_i(x_i) R_i^{-\frac{1}{2}}\| \sum_{j=1}^{N} \lambda_{ij} Q_j(x_j)^{\frac{1}{2}} \Big]$$

$$\therefore \dot{L}(x) \le -\xi^T \begin{bmatrix} \Theta & -\frac{1}{2}\Lambda^T \Theta \\ -\frac{1}{2}\Theta\Lambda & \Theta\Pi \end{bmatrix} \xi \triangleq -\xi^T A \xi \qquad (18)$$

where

$$\xi = \begin{bmatrix} Q_1(x_1)^{\frac{1}{2}} \\ Q_2(x_2)^{\frac{1}{2}} \\ \vdots \\ Q_N(x_N)^{\frac{1}{2}} \\ \|(\nabla J_1(x_1))^T g_1(x_1) R_1^{-\frac{1}{2}}\| \\ \|(\nabla J_2(x_2))^T g_2(x_2) R_2^{-\frac{1}{2}}\| \\ \vdots \\ \|(\nabla J_N(x_N))^T g_N(x_N) R_N^{-\frac{1}{2}}\| \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \ddots & \\ & & & \theta_N \end{bmatrix} \quad \Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{N1} & \lambda_{N2} & \cdots & \lambda_{NN} \end{bmatrix} \quad \Pi = \begin{bmatrix} \frac{\pi_1 - \frac{1}{2}}{2} & & & \\ & \frac{\pi_2 - \frac{1}{2}}{2} & & \\ & & \ddots & \\ & & & \frac{\pi_N - \frac{1}{2}}{2} \end{bmatrix}$$

$\exists\, \pi_i^*, \forall \pi_i \ge \pi_i^*$, let A positive definiteness. So $\left.\begin{array}{l} \dot{L}(x) < 0 \\ L(x) > 0 \end{array}\right\} \Rightarrow$ Lyapunov stability.

(Q.E.D.)

explore nonlinear subsystem using a known bounded piecewise continuous signal $e_i(t)$:

$$\dot{x}(t) = f_i(x_i(t)) + g_i(x_i(t))[u_i(x_i(t)) + e_i(t)]$$

$$\dot{V}_i(t) = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} = (\nabla V_i(x_i))^T[f_i(x_i) + g_i(x_i)u_i(x_i) + g_i(x_i)e_i]$$

$$V_i(x_i(t)) = \int_t^{\infty} r_i(x_i(\tau), u_i(\tau))d\tau \xrightarrow{\text{infinitestimal}} \frac{\partial v}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} = r_i(x_i, u_i) + (\nabla V_i(x_i))^T[f_i(x_i) + g_i(x_i)u_i(x_i)] = 0$$

$$\dot{V}_i(t) = -r_i(x_i, u_i) + (\nabla V_i(x_i))^T g_i(x_i)e_i$$

$$V_i^{(P)}(x_i(t+T)) - V_i^{(P)}(x_i(t)) = \int_t^{t+T} \dot{V}_i(x_i)d\tau = \int_t^{t+T} -r_i(x_i, u_i)d\tau + \int_t^{t+T}(\nabla V_i(x_i))^T g_i(x_i)e_i\, d\tau$$

$$= -\int_t^{t+T}[Q_i(x_i) + (u_i^{(P)}(x_i))^T R_i u_i^{(P)}(x_i)]d\tau + \int_t^{t+T} -2\left(-\tfrac{1}{2}\nabla V_i(x_i)\right)^T g_i(x_i)(R_i^{-1})^T R_i e_i\, d\tau$$

$$= -\int_t^{t+T}[Q_i(x_i) + (u_i^{(P)}(x_i))^T R_i u_i^{(P)}(x_i)]d\tau + \int_t^{t+T} -2\underbrace{\left[-\tfrac{1}{2}(R^{-1})g_i^T(x_i)\nabla V_i(x_i)\right]^T}_{u^{(P+1)}(x_i)} R\, e_i\, d\tau$$

$$= -2\int_t^{t+T}(u^{(P+1)}(x_i))^T R_i e_i\, d\tau - \int_t^{t+T}[Q_i(x_i) + (u_i^{(P)}(x_i))^T R_i u_i^{(P)}(x_i)]d\tau \qquad (25)$$

We obtain the online model-free integral PI algorithm.
without knowing the system dynamics: $f_i(x_i)$, $g_i(x_i)$

$$V_i^{(P)}(x_i(t)) = \int_t^{t+T}\underbrace{[Q_i(x_i) + (u_i^{(P)}(x_i))^T R_i u^{(P)}(x_i)]}_{r_i(x_i, u_i)}d\tau + 2\int_t^{t+T}(u_i^{(P+1)}(x_i))^T R_i e_i\, d\tau + V_i^{(P)}(x_i(t+T))$$