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RATIONAL INATTENTION WHEN DECISIONS TAKE TIME

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ABSTRACT

Decisions take time, and the time taken to reach a decision is likely to be informative about the cost of more precise judgments. We formalize this insight in the context of a dynamic rational inattention (RI) model. Under standard conditions on the flow cost of information in our discrete-time model, we obtain a tractable model in the continuous-time limit. We next provide conditions under which the resulting belief dynamics resemble either diffusion processes or processes with large jumps. We then demonstrate that the state-contingent choice probabilities predicted by our model are identical to those predicted by a static RI model, providing a micro-foundation for such models. In the diffusion case, our model provides a normative foundation for a variant of the DDM models studied in mathematical psychology.

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1 Introduction

It is common in economic modeling to assume that, when presented with a choice set, a decision maker (DM) will choose the option that is ranked highest according to a coherent preference ordering. However, observed choices in experimental settings often appear to be random, and while this could reflect random variation in preferences, it is often more sensible to view choice as imprecise. Models of rational inattention (such as Matêjka et al. [2015]) formalize this idea by assuming that the DM chooses her action based on a signal (a subjective assessment of the situation) that provides only an imperfect indication of the true state. The information structure that generates this signal is optimal, in the sense of allowing the best possible joint distribution of states and actions, net of a cost of information. In the terminology of Caplin and Dean [2015], models of rational inattention make predictions about patterns of state-dependent stochastic choice. These predictions will depend in part on the nature of the information cost, and several recent papers have attempted to recover information costs from observed behavior in laboratory experiments (Caplin and Dean [2015], Dean and Neligh [2019]).

However, in both laboratory experiments and real-world economic settings, decisions take time, and the time required to make a decision is likely to be informative about the nature of information costs. In this paper, we develop a framework to study rational inattention problems in which decisions take time, providing a means of connecting decision times to information costs and state-dependent stochastic choice.

There is an extensive literature in mathematical psychology that focuses on these issues. Variants of the drift-diffusion model (DDM, Ratcliff [1985], Ratcliff and Rouder [1998]) also make predictions about stopping times and state-dependent stochastic choice.¹ In particular, these models are designed to match the empirical observation that hasty decisions are likely to be of lower quality.² However, these models are not based on optimizing behavior, and this raises a question as to the extent to which they can be regarded as structural; it is unclear how the parameters of the DDM model should be expected to change when incentives or the costs of delay change, and this limits the use of the model for making counter-factual predictions. The framework we develop includes as a special case variants of the DDM model, while at the same time making predictions about state-dependent stochastic choice that match those of a static rational inattention model. Consequently, our framework is able to both speak to the relationship between stopping times and state-dependent stochastic choice (unlike standard rational inattention models) and make counter-factual predictions (unlike standard DDM models).

Specifically, we develop a class of rational inattention models in which the DM’s imprecise perception of the decision problem is not merely random, but evolves with the passage of time, and a constrained optimization problem determines a joint probability distribution over both stopping

¹DDM models were originally developed to explain imprecise perceptual classifications. See Woodford [2019] for a more general discussion of the usefulness of the analogy between perceptual classification errors and imprecision in economic decisions.

²The existence of a speed-accuracy trade-off is well-documented in perceptual classification experiments (e.g., Schouten and Bekker [1967]). Variants of the DDM that have been fit to stochastic choice data include Busemeyer and Townsend [1993] and more recently Krajbich et al. [2014] and Clithero [2018]; see Fehr and Rangel [2011] for a review of other early work. Shadlen and Shohamy [2016] provide a neural-process interpretation of sequential-sampling models of choice.

times and choices. We then demonstrate that the resulting state-dependent stochastic choice probabilities of our continuous-time model are equivalent to those of a static rational inattention model. Any cost function for a static rational inattention model in the uniformly posterior-separable family (in the terminology of Caplin et al. [2019]) can be justified in our framework. This result offers both a justification for using such cost functions in static rational inattention problem and provides a means of connecting those cost functions to dynamic processes for beliefs. Our results also provide a relationship between information costs in the static problem and decision times in the dynamic problem, and hence offer a new perspective on how the costs of information for rational inattention models might be uncovered from observable behavior.

Moreover, we provide conditions under which the process for beliefs in our model resembles a diffusion process, allowing our model to be interpreted as a variant of the DDM model. Our results therefore contribute to the literature on DDM-style models by offering an optimizing model that makes predictions about how decision boundaries and choice probabilities should change in response to changes in incentives, in particular by showing that these changes can be calculated using the comparative statics of a static rational inattention problem.

We begin by considering a discrete-time dynamic model of optimal information sampling, in which at each of the discrete time steps a random signal is obtained. We then develop our continuous-time framework as the limit of our discrete-time model when the length of the time steps between successive signals becomes small. In our discrete-time model, we assume that the information sampling process is optimally chosen from within a very flexible class of possibilities, in the spirit of models of rational inattention (Sims [2010]), with a flow information cost function that satisfies conditions commonly assumed in static models of rational inattention. The key step of our convergence proof is our demonstration that all such flow information cost functions can be approximated, for signal structures sufficiently close to uninformative, by a posterior-separable (in the terminology of Caplin et al. [2019]) flow information cost function.

Standard rational inattention models consider a static problem, in which a decision is made after a single noisy signal is obtained by the DM. This allows the set of possible signals to be identified with the set of possible decisions, which is no longer true in our dynamic setting. Steiner et al. [2017] also discuss a dynamic model of rational inattention, but their model is one in which, because of the kind of information cost assumed, it is never optimal to acquire information other than what is required for the current action. As a result, in each period of their discrete-time model, the set of possible signals can again be identified with the possible actions at that time. We instead consider situations in which evidence is accumulated over time before any action is taken, as in the DDM; this requires us to model the stochastic evolution of a belief state that is not simply an element of the set of possible actions.³ Our central concerns are to study the conditions under which the resulting continuous-time model of optimal information sampling gives rise to belief dynamics and stochastic choices similar to those implied by a DDM-like model, and to study how variations in the

³The model differs from the one analyzed by Steiner et al. [2017] in several key respects. First, as just noted, we study a setting in which the DM can take an action only once, and chooses when to stop and take an action endogenously. Second, we consider a much more general class of flow cost functions than the mutual information cost assumed in their paper. And third, we assume that the DM has a motive to smooth her information gathering over time, rather than learn all of the relevant information at a single point in time.

opportunity cost of time or the payoffs of actions should affect stochastic choice.

A number of prior papers have also sought to endogenize aspects of a DDM-like process. Moscarini and Smith [2001] consider both the optimal intensity of information sampling per unit of time and the optimal stopping problem, when the only possible kind of information is given by the sample path of a Brownian motion with a drift that depends on the unknown state, as assumed in the DDM.⁴ Fudenberg et al. [2018] consider a variant of this problem with a continuum of possible states, and an exogenously fixed sampling intensity.⁵ Woodford [2014] instead takes as given the kind of stopping rule posited by the DDM, but allows a very flexible choice of the information sampling process, as in theories of rational inattention. Our approach differs from these earlier efforts in seeking to endogenize *both* the nature of the information that is sampled at each stage of the evidence accumulation process and the stopping rule that determines how much evidence is collected before a decision is made.⁶

We place minimal restrictions on the nature of the flow information costs, and yet obtain relatively sharp conclusions about the nature of the continuous-time limit. While we allow in general for decisions about both stopping and information sampling to be arbitrary functions of the complete history of information sampled to that point, we show that it is possible to characterize the dynamics in terms of the evolution of the belief state in a finite-dimensional space, with stopping if and only if one of the stopping regions associated with the different possible actions is reached. We further show that under relatively weak conditions the dimensionality of the space in which beliefs move will be one less than the number of actions (thus, for example, a line in the case of a binary decision problem, as assumed in the DDM). Moreover, we show in general that the dynamics of the belief state prior to stopping can be described by a jump-diffusion process, and we give conditions under which it will be either a pure diffusion (as assumed in the DDM) or a pure jump process (as in the models of Che and Mierendorff [2019] and Zhong [2019]).

We also offer, under relative general conditions, a characterization of the boundaries of the stopping regions and the predicted ex ante probabilities of different actions, as functions of model parameters including the opportunity cost of time. The key to this characterization is a demonstration that in a broad class of cases, both the stopping regions and the ex ante choice probabilities for any given initial prior are the same as in a static RI problem with an appropriately chosen static information cost function. Thus in addition to providing foundations for interest in DDM-like models of the decision process, our paper provides novel foundations for interest in static RI problems of particular types. For example, we provide conditions under which the predictions of our model will be equivalent to those of a static RI model with the mutual-information cost function proposed by Sims [2010]) — and thus equivalent to the model of stochastic choice analyzed by Matějka et al. [2015] — but the foundations that we provide for this model do not rely on an analogy with rate-distortion

⁴Moscarini and Smith [2001] allow the instantaneous variance of the observation process to be freely chosen (subject to a cost), but this is equivalent to changing how much of the sample path of a given Brownian motion can be observed by the DM within a given amount of clock time.

⁵See also Tajima et al. [2016] for analysis of a related class of models, and Tajima et al. [2019] for an extension to the case of more than two alternatives.

⁶Both Morris and Strack [2019] and Zhong [2019] adopt our approach, and obtain special cases of the relationship between static and dynamic models of optimal information choice that we present below. Other recent dynamic models of optimal evidence accumulation include Che and Mierendorff [2019] and Zhong [2019], which differ from our treatment in not considering conditions under which beliefs will evolve as a diffusion process.

theory in communications engineering (the original motivation for the proposal of Sims).

More generally, we show that any cost function for a static RI model in the uniformly posterior-separable family studied by Caplin et al. [2019] can be justified by the process of sequential evidence accumulation that we describe. This includes the neighborhood-based cost functions discussed in Hébert and Woodford [2018], that lead to predictions that differ from those of the mutual-information cost function in ways that arguably better resemble the behavior observed in experiments such as those of Dean and Neligh [2019]. Our result provides both a justification for using such cost functions in static RI problems, and an answer (not given by static RI theory alone) to the question of how the cost function in an equivalent static RI model should change in the case of a change in the opportunity cost of time.

The connection that we establish between the choice probabilities implied by a dynamic model of optimal evidence accumulation and those implied by an equivalent static RI model holds both in the case that the belief dynamics in the dynamic model are described by a pure diffusion process and in the case that they are described by a jump process; thus we also show that with regard to these particular predictions, these two types of dynamic models are equivalent. However, this does not mean that no observations would make it possible to distinguish between them. We show that the predictions of the two types of model with regard to the distribution of decision times can be different, so that it should be possible in principle to use empirical evidence to determine which better describes actual decision making. And to the extent that we are interested in the predictions of our model for decision times and not solely for choices (as a number of authors have argued we should be), it matters which specification is the more realistic one.

We begin in section 2 by defining static RI problems, and introducing the discrete-time dynamic problem that we study and the continuous-time problem that is its limit. In section 3, we state the conditions that we impose on the flow information cost function, and present our result that all cost functions satisfying these conditions resemble posterior-separable cost functions in a local approximation. Section 4 demonstrates that the continuous-time model described in section 2 represents a limiting case of our discrete-time problem. In section 4 we also describe conditions under which beliefs follow a diffusion or a diffusion-like process, and conditions under which the belief dynamics instead involve large jumps. In section 5 we demonstrate that the state-dependent choice probabilities predicted by the continuous time models (in both the diffusion case and the jump case) are equivalent to those predicted by a static rational inattention model with a uniformly posterior-separable cost function. In section 6 we discuss how models involving jumps in beliefs and models involving diffusions can in principle be distinguished using stopping time data. In section 7 we conclude.

2 Static and Dynamic Models of Rational Inattention

We begin by describing the class of static rational inattention models surveyed by Sims [2010], and then describe the discrete and continuous time dynamic models we study.

Notation: given a set X , we define $\mathcal{P}(X)$ as the probability simplex associated with that set. We describe an element of the simplex $r \in \mathcal{P}(X)$ by a vector in $\mathbb{R}_+^{|X|}$ whose elements sum to one,

each of which corresponds to the likelihood of a particular element of $x \in X$. Except when necessary, we will call this vector r as well, and will not distinguish between the element of the simplex and its coordinate representation. We use the notation r_x to refer to the probability under r that $x \in X$ occurs. We discuss additional notation that appears in our proofs in the appendix, Section §B.

2.1 Static Models of Rational Inattention

Let $x \in X$ be the underlying state of nature, and let $s \in S$ be a signal the decision maker (DM) can receive, which might convey information about the state. We assume that X and S are finite sets. Let q denote the DM's prior belief (before receiving a signal) about the probability of state x ; that is, q is element of the probability simplex $\mathcal{P}(X)$. Define $p_{s,x}$ as the probability of receiving signal s in state x , let $p_x \in \mathcal{P}(S)$ be the associated conditional probability distribution of the signals given state x , and let p be the $|S| \times |X|$ matrix whose elements are $p_{s,x}$. The matrix p , which is a set of conditional probability distributions for each state of nature, $\{p_x\}_{x \in X}$, defines an ‘‘information structure.’’

By Bayes' rule, the DM believes under her prior that the unconditional probability of receive a signal $s \in S$ is $\pi_s(p, q) = \sum_{x \in X} p_{s,x} q_x$. After receiving signal s , the DM will hold a posterior, $q_s(p, q) \in \mathcal{P}(X)$, which is defined by

$$q_{s,x}(p, q) = \frac{p_{s,x} q_x}{\pi_s(p, q)}$$

for all $s \in S$ such that $\pi_s(p, q) > 0$. We adopt the convention that $q_s(p, q) = q$ for all $s \in S$ such that $\pi_s(p, q) = 0$.

Let $a \in A$ be the action taken by the decision maker (DM). For simplicity, A is also a finite set, and we assume that the number of states is weakly larger than the number of actions, $|X| \geq |A|$. The DM's utility from taking action a in state x is $u_{a,x}$. We assume that $u_{a,x}$ is strictly positive and bounded above by a positive constant, \bar{u} .

The maximum achievable expected payoff, given an information structure p and prior q , can be written as

$$u(p, q) \equiv \max_{\{a(s)\}} \sum_{x \in X} \sum_{s \in S} q_x p_{s,x} u_{a(s),x}. \quad (1)$$

The standard static rational inattention problem, given the signal alphabet S ,⁷ is then

$$\max_{\{p_x \in \mathcal{P}(S)\}_{x \in X}} u(p, q) - C(p, q; S), \quad (2)$$

where

$$C(\cdot, \cdot; S) : \mathcal{P}(S)^{|X|} \times \mathcal{P}(X) \rightarrow \mathbb{R} \quad (3)$$

is a cost function for information structures.

In the classic formulation of Sims, a problem of the form equation (2) is considered, in which the

⁷The full problem includes a choice over the signal alphabet S . A standard result, which will hold for all of the cost functions we study, is that $|S| = |A|$ is sufficient.

cost function $C(p, q; S)$ is proportional to the Shannon mutual information between the signal and the state. Mutual information can be defined using Shannon’s entropy,

$$H^{Shannon}(q) \equiv - \sum_{x \in X} q_x \ln(q_x). \quad (4)$$

Shannon’s entropy can in turn be used to define a measure of the degree to which the posterior $q' \in \mathcal{P}(X)$ associated with any signal differs from the prior q , the Kullback-Leibler (KL) divergence,

$$D_{KL}(q' || q) \equiv H^{Shannon}(q) - H^{Shannon}(q') + (q' - q)^T \cdot H_q^{Shannon}(q), \quad (5)$$

where $H_q^{Shannon}(q)$ denotes the gradient of Shannon’s entropy evaluated at beliefs q . Mutual information is then the expected value of the KL divergence over possible signals, and the cost of an information structure is assumed to be proportional to the mutual information,

$$C^{MI}(p, q; S) = \theta \sum_{s \in S} \pi_s(p, q) D_{KL}(q_s(p, q) || q), \quad (6)$$

where $\theta > 0$ is a positive constant. Mutual information is a measure of the informativeness of the signal, in that it provides a measure of the degree to which the signal changes what one should believe about the state, on average. However, Shannon’s mutual information is not the only possible measure of the informativeness of an information structure, or the only plausible cost function for a static rational inattention problem. We discuss two large classes of cost functions, both of which will feature prominently in our analysis, below. For specific alternative proposals, see Caplin et al. [2019], Hébert and Woodford [2018], Pomatto et al. [2018].

Caplin et al. [2019] define a large class of cost functions, the “posterior-separable” class, which can be written as

$$C^{PS}(p, q; S) = \sum_{s \in S} \pi_s(p, q) D(q_s(p, q) || q), \quad (7)$$

where $D(\cdot || \cdot)$ is a divergence. Recall that a divergence is defined as a function $D : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}_+$, with $D(q' || q) = 0$ if and only if $q' = q$. Divergences can be thought of as distances between probability distributions, although they are not necessarily symmetric and do not necessarily satisfy the triangle inequality. Mutual information is a posterior-separable cost function, whose associated divergence is (proportional to) the KL divergence.

Within this class, a cost function is said to be “uniformly posterior-separable” if the divergence D is a Bregman divergence, meaning that

$$D(q' || q) = H(q') - H(q) - (q' - q)^T \cdot H_q(q) \quad (8)$$

for some convex function $H : \mathcal{P}(X) \rightarrow \mathbb{R}$. The KL divergence is a Bregman divergence, whose associated convex function is the negative of Shannon’s entropy. These two classes, the posterior-separable cost functions and the uniformly posterior-separable cost functions, will play a key role in our analysis.

In our derivation of the continuous time limit of our discrete time model, we adopt a more general approach (following De Oliveira et al. [2017] and Caplin and Dean [2015]) and study the entire class of costs functions satisfying certain conditions. We introduce these conditions after describing our discrete and continuous time dynamic models.

2.2 Dynamic Discrete Time Models of Rational Inattention

We next extend the static rational inattention model just described to a discrete time dynamic setting, in which the DM has numerous opportunities to gather information before taking an action. Our extension is similar in spirit to the one described by Steiner et al. [2017], with several significant differences. First, unlike those authors, we study a setting in which the DM can take an action only once, and chooses when to stop and take an action endogenously. Second, we are interested in general cost functions of the form described by equation (3), not just mutual information. Third, we assume the DM has a motive to smooth her information gathering over time, rather than learn all of the relevant information in a single period.

As in the static model, there is a state of the world, $x \in X$, that remains constant over time. At each time t , the DM can either stop and take an action $a \in A$, or continue and receive a signal drawn from the information structure $\{p_{t,x} \in \mathcal{P}(S)\}_{x \in X}$, for some signal alphabet S . We assume that the signal alphabet S is finite and fixed over time, with $|S| \geq 2|X|+2$. However, the information structure $\{p_{t,x}\}_{x \in X}$ is a choice variable that can depend on the past history of signal realizations. Fixing the signal alphabet S has no economic meaning, because the information content of receiving a particular signal $s \in S$ can change over time; our assumption that S is finite allows us to avoid technicalities associated with infinite-dimensional signal structures. We also assume, as a technical device, that S contains one signal, \bar{s} , that is required to be uninformative. This assumption ensures that the DM can choose to mix any arbitrary signal structure with an uninformative one, even if she has already used up her “useful” signals.

The DM’s prior beliefs at time t , before receiving the signal, are denoted q_t . Each time period has a length Δ . Let τ denote the time at which the DM stops and makes a decision, with $\tau = 0$ corresponding to making a decision without acquiring any information. At this time, the DM receives utility $u_{a,x} - \kappa\tau$ if she takes action a at time τ and the true state of the world is x . Let $\hat{u}(q_\tau)$ be the utility (not including the penalty for delay) associated with taking an optimal action under beliefs q_τ :

$$\hat{u}(q_\tau) = \max_{a \in A} \sum_{x \in X} q_{\tau,x} u_{a,x}.$$

We assume, as in the static model, that $u_{a,x}$ is strictly positive and bounded above by the constant \bar{u} , and that $|A| \leq |X|$.

Each time period has length Δ , and the DM discounts the future exponentially, with discount factor β^Δ , for some $\beta \in (0, 1]$. The parameter κ and discount factor β together govern the size of the penalty the DM faces from delaying his decision. The DM does not always make a decision immediately because she is able to gather information over time before making a more-informed decision.

The DM can choose an information structure that depends on the current time and past history of the signals received. As we will see, the problem has a Markov structure, and the current time’s “prior,” q_t , summarizes all of the relevant information that the DM needs to design the information structure. The DM is constrained to satisfy

$$E_0\left[\frac{\Delta}{\rho} \sum_{j=0}^{\tau\Delta^{-1}-1} \beta^{j\Delta} C(p_{j\Delta}, q_{j\Delta}; S)^\rho\right]^{\frac{1}{\rho}} \leq \Delta c E_0\left[\Delta \sum_{j=0}^{\tau\Delta^{-1}-1} \beta^{j\Delta}\right]^{\frac{1}{\rho}}, \quad (9)$$

if the DM choose to acquire any information at all ($\tau > 0$ always in this case).⁸ In words, the L^ρ -norm of the flow information cost function $C(\cdot)$ over time and possible histories must be less than the constant c per unit time. In the particular case in which information gathering is constant ($C(p_{j\Delta}, q_{j\Delta}; S) = \bar{C}$), this constraint simplifies to

$$\frac{\bar{C}}{\Delta} \leq \rho^{\frac{1}{\rho}} c,$$

emphasizing that the constant c can be thought of as a limit on the flow rate of information acquisition.

The parameter ρ governs the substitutability of information acquisition across time. In the limit as $\rho \rightarrow \infty$, the L^ρ norm becomes the essential supremum, and the constraint approaches a per-period constraint on the amount of information the DM can obtain. For finite values of ρ , the DM can allocate more information gathering to states and times in which it is more advantageous to gather more information. We assume, however, that $\rho > 1$, to ensure that it is optimal for the DM to gather information gradually, rather than all at once. Our assumption of $\rho > 1$ is similar to the convex cost of the rate of experimentation assumed by Moscarini and Smith [2001]. It is a critical assumption that separates our model from the model of Steiner et al. [2017], in which the DM will (under some circumstances) learn a large amount of information in a single period. (Further assumptions about the flow cost function $C(\cdot)$ are introduced in the next section.)

Let $V(q_0; \Delta)$ denote the value obtained in the sequence problem for a DM with prior beliefs q_0 , and let q_τ denote the DM’s beliefs when stopping to act. The DM’s problem is

$$V(q_0; \Delta) = \max_{\{p_{j\Delta}\}, \tau} E_0\left[\beta^\tau \hat{u}(q_\tau) - \Delta \kappa \frac{1 - \beta^\tau}{1 - \beta^\Delta}\right], \quad (10)$$

subject to the information-cost constraint (9).

Note that the constraint (9) can equivalently be written in a form that takes each side of the inequality to the power ρ ; we then obtain a constraint that is additively separable over time and across possible histories. This allows us to write a Lagrangian for the DM’s problem in the additively

⁸The factor Δ inside the operator $E_0[\cdot]$ on both sides of this inequality might seem redundant. We include it so that the expression inside the operator on the left-hand side approaches a time integral of discounted information costs in the limit as Δ is made arbitrarily small.

separable form

$$W(q_0, \lambda; \Delta) = \max_{\{p_{j\Delta}\}, \tau} E_0[\beta^\tau \hat{u}(q_\tau) - \Delta \kappa \frac{1 - \beta^\tau}{1 - \beta\Delta}] - \lambda E_0[\Delta^{1-\rho} \sum_{j=0}^{\tau\Delta^{-1}-1} \beta^{j\Delta} \{\frac{1}{\rho} C(p_{j\Delta}, q_{j\Delta}; S)^\rho - \Delta^\rho c^\rho\}], \quad (11)$$

where $\lambda \geq 0$ is a Lagrange multiplier associated with the information-cost constraint. With this definition, the value function defined in (10) corresponds to $V(q_0; \Delta) = \min_{\lambda \geq 0} W(q_0, \lambda; \Delta)$. An optimal information sampling policy then necessarily maximizes $W(q_0, \lambda; \Delta)$ for some value of $\lambda \geq 0$. This alternative formulation of the problem can be thought of as the value function of a different problem, in which there is a cost of gathering information proportional to $\lambda \frac{1}{\rho} C(\cdot)^\rho$.

We will next analyze the continuous time limits of these functions,

$$W(q_0, \lambda) = \lim_{\Delta \rightarrow 0^+} W(q_0, \lambda; \Delta)$$

and

$$V(q_0) = \lim_{\Delta \rightarrow 0^+} V(q_0; \Delta).$$

We will first present the limiting continuous time problem, and then state assumptions that ensure this continuous time problem is the limit of the discrete time problem. Our approach is to assume standard conditions on the flow cost function $C(\cdot)$, which we describe in the next section, and then characterize the cost of a small amount of information under these conditions. We demonstrate that, to an approximation of first order in Δ , every cost function satisfying our conditions is equivalent to a posterior-separable cost function. Because of our assumption that $\rho > 1$, it will be optimal, as $\Delta \rightarrow 0^+$, for the DM to acquire a small amount of information each period, and consequently our results about the cost of a small amount of information will determine the form of the continuous time limit.

2.3 Dynamic Continuous Time Models of Rational Inattention

Here we introduce the continuous time models that are the limits of our discrete time model. We will discuss two cases: a model with exponential discounting ($\beta < 1$) and a model with no discounting ($\beta = 1$). In both of these models, beliefs follow a jump-diffusion process. In our analysis of the models (section §4), we will discuss conditions under which optimal information sampling implies that beliefs follow pure jump or pure diffusion processes.

Our continuous time models are defined in terms of a divergence, $D^*(r||q)$. This divergence is continuously twice-differentiable in r for r sufficiently close to q , and we define an $|X| \times |X|$ positive semi-definite matrix-valued function $k(q)$ by

$$\frac{\partial^2 D^*(r||q)}{\partial r^i \partial r^j} \Big|_{r=q} = \text{Diag}(q) k(q) \text{Diag}(q), \quad (12)$$

where $\text{Diag}(q)$ is an $|X| \times |X|$ diagonal matrix whose diagonal is the vector q . We call this $k(q)$

function the “information cost matrix” and we prove it satisfies certain properties as part of our derivation, which we describe in section §4.

The key step in our derivation of the continuous time model is Theorem 1, which demonstrates that, up to first-order in Δ , every cost function satisfying our conditions is posterior-separable:

$$C(p, q; S) = \Delta \sum_{s \in S} \pi_s(p, q) D^*(q_s(p, q) || q) + o(\Delta) \quad (13)$$

for all sufficiently uninformative p (for a more formal statement, see Theorem 1 below). Loosely, there are two ways for a signal structure to be close to uninformative. First, it might have signals that occur frequently but whose associated posteriors are close to the value of the prior. Second, it might have signals that occur rarely but whose posteriors are far from the prior. These two kinds of signals correspond in the continuous time limit, respectively, to a diffusion component and a jump component of the stochastic process for beliefs.⁹

We prove in the limit as $\Delta \rightarrow 0^+$ that the process q_t converges to a jump-diffusion:

$$dq_t = -\bar{\psi}_t z_t dt + z_t dJ_t + \text{Diag}(q_t) \sigma_t dB_t,$$

where dJ_t is a poisson process with intensity $\bar{\psi}_t$, z_t is change in beliefs conditional on a jump, dB_t is an $|X|$ -dimensional Brownian motion, and σ_t is an $|X| \times |X|$ matrix that controls the diffusion component of beliefs. Because beliefs must remain in the simplex, $q_{t-} + z_t$ is absolutely continuous with respect to q_{t-} , and $q_t^T \sigma_t = \vec{0}$.

The flow information cost in this limit is

$$C(\sigma_t, \bar{\psi}_t, z_t; q_t) dt = \frac{1}{2} \text{tr}[\sigma_t \sigma_t^T k(q_t)] dt + \bar{\psi}_t D^*(q_t + z_t || q_t) dt,$$

which the continuous time analog of equation (13).

We now introduce the continuous time analog of the (dual) dynamic discrete time problem (equation (11)), which we will refer to as $W^+(q_t, \lambda)$.

Definition 1. The dual continuous time problem with discounting ($\beta < 1$) is

$$W(q_t, \lambda) = \sup_{\{\sigma_s \in \mathbb{R}^{|X| \times |X|}, \bar{\psi}_s \in \mathbb{R}_+, z_s \in \mathbb{R}^{|X|}, \tau \in \mathbb{R}_+\}} E_t[\beta^{(\tau-t)} \hat{u}(q_\tau) - \frac{1}{-\ln(\beta)} (1 - \beta^{(\tau-t)}) (\kappa - \lambda c^\rho)] - \frac{\lambda}{\rho} E_t[\int_t^\tau \beta^{(s-t)} \{ \frac{1}{2} \text{tr}[\sigma_s \sigma_s^T k(q_s)] + \bar{\psi}_s D^*(q_{s-} + x_s || q_{s-}) \}^\rho ds],$$

subject to the evolution of beliefs,

$$dq_t = -\bar{\psi}_t x_t dt + x_t dJ_t + \text{Diag}(q_t) \sigma_t dB_t,$$

⁹Signals whose frequency converges to zero and whose posterior converges to the prior can in fact be informative, if they converge at slow enough rates. When the frequency does not converge to zero, the signal is a diffusion in the continuous time limit. When the posterior does not converge to a prior, the signal is a jump in the continuous time limit. When both converge but sufficiently slowly, the signal becomes a non-jump-diffusion semi-martingale in the continuous time limit. However, part of our convergence proof demonstrates that it is without loss of generality to assume beliefs follow a jump-diffusion process.

where dJ_t is a Poisson process with intensity $\bar{\psi}_t$ and dB_t is an $|X|$ -dimensional Brownian motion, and the constraints that $q_{t-} + x_t \ll q_{t-}$ and $q_t^T \sigma_t = \vec{0}$, and the constraint that, for all stopping times T measurable with respect filtration generated by q_s^* ,

$$E_t \left[\int_t^T \beta^{(s-t)} \left\{ \frac{1}{2} \text{tr}[\sigma_s \sigma_s^T k(q_t)] + \bar{\psi}_s D^*(q_{s-} + x | q_{s-}) \right\} ds \right] \leq \left(\frac{\theta}{\lambda} \right)^{\frac{1}{\rho-1}} E_t \left[\frac{1 - \beta^{(T-t)}}{-\ln(\beta)} \right],$$

where θ is a positive constant.

For the problem with discounting, there exists some λ^* such that the limit of the original sequence problem, $\lim_{n \rightarrow \infty} V(q_0; \Delta_n) = V(q_0)$, is equal to $W^+(q_t, \lambda^*)$.

In the particular case where the DM has no exponential discounting ($\beta = 1$), we demonstrate that the amount of information acquired at each moment ($C(\sigma_t, \psi_t, x_t; q_t)$ above) is constant. This property comes from the fact that the cost of delay is constant. In the case with discounting ($\beta < 1$), the cost of delay depends in part on the current level of the value function, which generates variation and leads to a non-constant flow cost function. In the $\beta = 1$ case, using the fact that the quantity of information acquired is constant, the problem can be written in a simpler form. We introduce this simpler form below, and explicitly characterize both the dual continuous time problem and the original sequence problem.

Definition 2. The dual continuous time problem without discounting ($\beta = 1$) is, for all $\lambda \in (0, \kappa c^{-\rho})$,

$$W(q_t, \lambda) = \sup_{\{\sigma_s \in \mathbb{R}^{|X| \times |X|}, \bar{\psi}_s \in \mathbb{R}_+, z_s \in \mathbb{R}^{|X|}, \tau \in \mathbb{R}_+\}} E_t[\hat{u}(q_{\tau^*}) - (\tau - t) \frac{\rho}{\rho - 1} (\kappa - \lambda c^\rho)]$$

subject to the constraints that $q_{t-} + z_t \ll q_{t-}$, $q_t^T \sigma_t = \vec{0}$, and

$$\frac{1}{2} \text{tr}[\sigma_s \sigma_s^T k(q_t)] + \bar{\psi}_s D^*(q_{s-} + z_s | q_{s-}) \leq \chi(\lambda),$$

for all times $s \in [t, \tau)$ and some constant $\chi(\lambda) > 0$.

The limit of the original sequence problem, $\lim_{n \rightarrow \infty} V(q_0; \Delta_n) = W(q_t, \lambda^*) = V(q_0)$, is

$$V(q_t) = \sup_{\{\sigma_s \in \mathbb{R}^{|X| \times |X|}, \bar{\psi}_s \in \mathbb{R}_+, z_s \in \mathbb{R}^{|X|}, \tau \in \mathbb{R}_+\}} E_t[\hat{u}(q_{\tau^*}) - (\tau - t) \kappa]$$

subject to the constraints that $q_{t-} + z_t \ll q_{t-}$, $q_t^T \sigma_t = \vec{0}$, and

$$\frac{1}{2} \text{tr}[\sigma_s \sigma_s^T k(q_t)] + \bar{\psi}_s D^*(q_{s-} + z_s | q_{s-}) \leq \rho^{\frac{1}{\rho}} c.$$

These continuous time problems are described using the “unconditional” dynamics for the beliefs q_t , meaning that beliefs are martingales. Using Bayes’ rule, we can also define the conditional dynamics for beliefs. Suppose that $x_{true} \in X$ is the true state of nature. The beliefs q_t follow the

dynamics process

$$dq_t = \text{Diag}(q_t)\sigma_t\sigma_t^T e_{x_{true}} - \bar{\psi}_t z_t dt + x_t dJ_t^{x_{true}} + \text{Diag}(q_t)\sigma_t dB_t, \quad (14)$$

where $e_{x_{true}}$ is a vector equal to one in the state corresponding to x_{true} and zero otherwise, and $dJ_t^{x_{true}}$ is a Poisson process with arrival rate $\bar{\psi}_t(1 + \frac{e_{x_{true}}^T z_t}{e_{x_{true}}^T q_t})$. Intuitively, beliefs will tend to move towards x_{true} . Consider in particular the value $e_{x_{true}}^T dq_t$. There is a positive (by the positive definiteness of $\sigma_t\sigma_t^T$) drift term associated with the diffusion of beliefs, and jumps are more likely if $e_{x_{true}}^T z_t$ is positive and less likely if it is negative. Certain other models in the literature (e.g. DDM models, see Fudenberg et al. [2018]) are usually expressed in terms of the conditional dynamics of beliefs, and equation (14) allows us to relate our continuous time models to these other models.

Before proceeding to our derivation, we briefly discuss the differences between the $\beta < 1$ and $\beta = 1$ cases, and of the role of jumps vs. diffusion. Beginning with the issue of jumps vs. diffusions, observe that there is a kind of continuity between the two. Because the cost of both jumps and diffusion in beliefs are determined by the same divergence, D^* , there exists a limit in which the jump becomes very likely ($\bar{\psi}_t \rightarrow \infty$) and very small ($z_t \rightarrow 0$) and for which the stochastic process of beliefs and the cost of the information converge to the stochastic process and cost of a diffusion process. That is, it is without loss of generality in both our continuous time problems to assume that $\sigma_t = 0$, because the supremum of a sequence of jump controls $(\bar{\psi}_t, z_t)$ can perfectly replicate a diffusion process. Relatedly, although both of these problems appear to allow the DM to have jumps in only a single direction z_t , nothing requires that the DM choose controls $(\bar{\psi}_t, z_t)$ that vary smoothly in the state q_t . Consequently, the DM is perfectly able to replicate a process with jumps in multiple directions. These statements are proven as part of the proof of our convergence result, Theorem 2 below.

Turning now to the issue of $\beta < 1$ vs. $\beta = 1$, observe first that many decisions are made over short periods of time (seconds or minutes). With conventional rates of time preference, β should be extremely close to one. As we will demonstrate, in the $\beta = 1$ case, the model is tractable and we are able (under certain additional assumptions) to characterize the value function. Consequently, provided that behavior is continuous in the limit as β approaches one (and we will show that it is), we believe it is reasonable to focus on the $\beta = 1$ model.

Moreover, as noted by Fudenberg et al. [2018], when $\beta < 1$, the dynamics of beliefs are not invariant to non-action-contingent transformations of the utility function. That is, increasing the utility function will change the DM's behavior, because with $\beta < 1$ a higher value function creates a higher cost of delay. We view this as undesirable, and describe in Section §6 an experimentally feasible test of this implication of $\beta < 1$. This association between the cost of delay and the level of the value function is the key property that leads to the result of Zhong [2019] that, in a special case of our setting, jumps in beliefs and not diffusions are optimal. This result might lead one to conclude that there is a discontinuity between the $\beta < 1$ and $\beta = 1$ cases. However, we show below that continuity holds between the $\beta < 1$ and $\beta = 1$ cases. That is, in situations in which the $\beta = 1$ model leads to a diffusion process for beliefs, the $\beta < 1$ model has beliefs that are a sequence of tiny jumps, whose magnitude is on the order of $-\ln(\beta)$, and the belief process converges in the limit to

a diffusion.

Having introduced static, discrete time dynamic, and continuous time dynamic models of rational inattention, we next outline the key conditions we impose on flow cost functions in the discrete time model that lead the continuous time model as the limit.

3 Flow Information Costs

At each date in the discrete-time sequential rational inattention problem introduced above, the DM chooses an information structure. Each information structure p has a flow cost $C(p, q; S)$, given by a function of the form of (3), where q is the DM's prior in this date (that is, the posterior beliefs following from observations prior to the current date of the dynamic problem), and S is the signal alphabet.¹⁰ Our results depend only on assuming that this flow information-cost function satisfies a set of five general conditions, stated below.

All of these conditions are satisfied by the mutual-information cost function (6) proposed by Sims, but they are also satisfied by many other cost functions (for example, the neighborhood-based cost functions we describe in Hébert and Woodford [2018]). They are closely related to conditions that other authors (De Oliveira et al. [2017] and Caplin and Dean [2015]) have also proposed as attractive general properties to assume about information-cost function, in the context of static rational inattention models.

Condition 1. Information structures that convey no information ($p_x = p_{x'}$ for all x, x' in the support of q) have zero cost. All other information structures have a strictly positive cost.

This condition ensures that the least costly strategy for the DM in the standard static rational inattention problem is to acquire no information, and make her decision based on the prior. The requirement that gathering no information has zero cost is a normalization.

The next condition is called mixture feasibility by Caplin and Dean [2015]. Consider two information structures, $\{p_{1,x}\}_{x \in X}$, with signal alphabet S_1 , and $\{p_{2,x}\}_{x \in X}$, with alphabet S_2 . Given a parameter $\lambda \in (0, 1)$, we define a mixed information structure, $\{p_{M,x}\}_{x \in X}$ over the signal alphabet $S_M = (S_1 \cup S_2) \times \{1, 2\}$. For each $s = (s_1, 1)$ in the alphabet S_M , $p_{M,x}(s)$ is equal to $\lambda p_{1,x}(s)$ if $s_1 \in S_1$, and equal to 0 otherwise. Likewise, for each $s = (s_2, 2)$, $p_{M,x}(s)$ is equal to $(1 - \lambda)p_{2,x}(s)$ if $s_2 \in S_2$, and equal to 0 otherwise.

That is, this information structure results, with probability λ , in a posterior associated with information structure p_1 , and with probability $1 - \lambda$ in a posterior associated with information structure p_2 . The distribution of posteriors under the mixed information structure is a convex combination of the distributions of posteriors under the two original information structures, as if the DM flipped a coin and used the result to chose one of the two information structures, and (crucially) remembered the result of the coin flip. The mixture feasibility condition requires that choosing a

¹⁰The information-cost functions that we study, like mutual information, are defined for all finite signal alphabets S . Note, however, that mutual information is also defined over alternative sets of states of nature X . We do not impose this requirement on our more general cost functions — all of our analysis takes the set of states of nature as given.

mixed information structure costs no more than the cost of randomizing over information structures (using a mixed strategy in the rational inattention problem).

Condition 2. Given two information structures, $\{p_{1,x}\}_{x \in X}$, with signal alphabet S_1 , and $\{p_{2,x}\}_{x \in X}$, with alphabet S_2 , the cost of the mixed information structure is weakly less than the weighted average of the cost of the separate information structures:

$$C(p_M, q; S_M) \leq \lambda C(p_1, q; S_1) + (1 - \lambda)C(p_2, q; S_2).$$

The next condition uses Blackwell's ordering. Consider two signal structures, $\{p_x\}_{x \in X}$, with signal alphabet S , and $\{p'_x\}_{x \in X}$, with alphabet S' . The first information structure Blackwell dominates the second information structure if, for all utility functions $u_{a,x}$ and all priors $q \in \mathcal{P}(X)$,

$$u(p, q) \geq u(p', q),$$

where $u(p, q)$ is defined as in equation (1). In words, if one information structure Blackwell dominates another, it is weakly more useful for every decision maker, regardless of that decision maker's utility function and prior. In this sense, it conveys weakly more information. This ordering is incomplete; most information structures neither dominate nor are dominated by a given alternative information structure. However, when an information structure does Blackwell dominate another one, we assume that the dominant information structure is weakly more costly.

Condition 3. If the information structure $\{p_x\}_{x \in X}$ with signal alphabet S is more informative, in the Blackwell sense, than $\{p'_x\}_{x \in X}$, with signal alphabet S' , then, for all $q \in \mathcal{P}(X)$,

$$C(\{p_x\}_{x \in X}, q; S) \geq C(\{p'_x\}_{x \in X}, q; S').$$

The first three conditions are, from a certain perspective, almost innocuous. For any joint distribution of actions and states that could have been generated by a DM solving a static rational inattention problem, with an arbitrary information cost function, there is a cost function consistent with these three conditions that also could have generated that data (Theorem 2 of Caplin and Dean [2015]). The result arises from the possibility of the DM pursuing mixed strategies over information structures and mixed strategies in the mapping between signals and actions. These conditions also characterize "canonical" rational inattention cost functions, in the terminology of De Oliveira et al. [2017].

We demonstrate that the mixture feasibility condition (Condition 2) and Blackwell monotonicity condition (Condition 3) are equivalent to requiring that the cost function be convex over information structures and Blackwell monotone.

Lemma 1. *Let p and p' be information structures with signal alphabet S . A cost function is convex in information structures if, for all $\lambda \in (0, 1)$, all signal alphabets S , and all $q \in \mathcal{P}(X)$,*

$$C(\lambda p + (1 - \lambda)p', q; S) \leq \lambda C(p, q; S) + (1 - \lambda)C(p', q; S).$$

A cost function satisfies mixture feasibility and Blackwell monotonicity (Conditions 2 and 3) if and only if it is convex in information structures and satisfies Blackwell monotonicity.

Proof. See the appendix, section B.1. □

This result is useful because it allows us to restrict attention to cost functions that are convex in the signal structure, which is the choice variable at each date.

The fourth condition that we assume, which is not imposed by Caplin and Dean [2015], Caplin et al. [2019], or De Oliveira et al. [2017], is a differentiability condition that will allow us to characterize the local properties of our cost functions. Let $p^0(p, q)$ be an uninformative signal structure, with $p_{x,s}^0(p, q) = \pi_s(p, q)$ for all $x \in X$ and $s \in S$. That is, for each state $x \in X$, with the signal structure p^0 , the likelihood of receiving each signal $s \in S$ is equal to the unconditional likelihood, and hence the signal is completely uninformative. Treating p and $p^0(p, q)$ as elements of $\mathbb{R}^{|X| \times |S|}$, let $\|p - p^0(p, q)\|$ be an arbitrary norm on the difference between two signal structures.

Condition 4. There exists an $\epsilon > 0$ such that, for all signal alphabets S and priors q and all information structures p sufficiently close to uninformative ($\|p - p^0(p, q)\| < \epsilon$), the information cost function is continuously twice-differentiable with respect to p in all directions that do not change the support of the signal distribution, and directionally differentiable, with a continuous directional derivative, with respect to perturbations that increase the support of the signal distribution.

While this may seem a relatively innocuous regularity condition, it is not completely general; for example, it rules out the case in which the DM is constrained to use only signals in a parametric family of probability distributions, and the cost of other information structures is infinite. Thus it rules out information structures of the kind assumed in Fudenberg et al. [2018] or Morris and Strack [2019]. Condition 4 also rules out other proposed alternatives, such as the channel-capacity constraint suggested by Woodford [2012]. The “ $\epsilon > 0$ ” part of the condition indicates that that this differentiability need only hold at nearly uninformative information structures; obviously, if differentiability holds everywhere, the condition will be satisfied.

We should also note that this condition imposes the assumption that the cost function is finite for posteriors on the boundary of the simplex. This feature of the condition is convenient but unnecessary for our results; we could weaken the condition to require directional differentiability with respect to perturbations that increase the support of the signal distribution only for perturbations for which the resulting posteriors are interior to the simplex defined by the support of q . We should also observe that, because the convex function is convex in p , the existence of a directional derivative is guaranteed. What the condition is imposing is that this directional derivative be continuous.

The next condition that we assume, which is also not imposed by Caplin and Dean [2015], Caplin et al. [2019], or De Oliveira et al. [2017], is a sort of local strong convexity. We will assume that the cost function exhibits strong convexity, in the neighborhood of an uninformative information structure, with respect to information structures that hold fixed the unconditional distribution of signals, uniformly over the set of possible priors.

Condition 5. There exists constants $m > 0$ and $B > 0$ such that, for all priors $q \in \mathcal{P}(X)$, and all information structures that are sufficiently close to uninformative ($C(p, q; S) < B$),

$$C(p, q; S) \geq \frac{m}{2} \sum_{s \in S} \pi_s(p, q) \|q_s(p, q) - q\|_X^2,$$

where q_s is the posterior given by Bayes' rule and $\|\cdot\|_X$ is an arbitrary norm on the tangent space of $\mathcal{P}(X)$.

This condition is slightly stronger than Condition 1; it is essentially an assumption of “local strong convexity” instead of merely local strict convexity. It implies that all informative information structures have a non-trivial positive cost, and that (regardless of the DMs' current beliefs) there are no informative information structures that are “almost free.” This condition allows us to assert that if the flow cost $C(p, q; S)$ is converging to zero, then either the posteriors must become close to the prior ($q_s(p, q)$ close to q) or the signals must become rare ($\pi_s(p, q)$ close to zero), or some combination thereof.

The mutual-information cost function (6) satisfies each of these five conditions. However, it is not the only cost function to do so. For example, we can construct a family of such cost functions, using the family of “f-divergences” (see, e.g., Ali and Silvey [1966] or Amari and Nagaoka [2007]), defined as

$$D_f(q' \| q) = \sum_{x \in X} q_x f\left(\frac{q'_x}{q_x}\right),$$

where f is any strictly convex, twice-differentiable function with $f(1) = f'(1) = 0$ and $f''(1) = 1$. The KL divergence is a member of this family, corresponding to $f(u) = u \ln u - u + 1$.

The posterior-separable cost function associated with this divergence,

$$C^f(p, q; S) = \sum_{s \in S} \pi_s(p, q) D_f(q_s(p, q) \| q), \tag{15}$$

satisfy all five of the conditions described above. Another example satisfying our conditions, as mentioned previously, are the neighborhood cost functions of described by Hébert and Woodford [2018]. These cost functions fall into the uniformly posterior-separable family of cost functions described by Caplin et al. [2019], and (under mild regularity assumptions) all such functions satisfy our conditions. We discuss these issues in more detail in the appendix, section §A.

We are now in a position to use these conditions to characterize the cost of a small amount of information.

Theorem 1. *Let Δ_m , $m \in \mathbb{N}$, denote a sequence such that $\lim_{m \rightarrow \infty} \Delta_m = 0$. Given an prior $q \in \mathcal{P}(X)$ and signal alphabet S , let $p_m \in \mathcal{P}(S)^{|X|}$ be a sequence of signal structures, and suppose that there exists a $B > 0$ such that, for all $s \in S$ and all $m \in \mathbb{N}$,*

$$\pi_s(p_m, q) \|q_s(p_m, q) - q\|_X^2 \leq B \Delta_m.$$

Then any cost function satisfying Conditions 1-4 is posterior-separable up to first order in Δ :

$$C(p_m, q; S) = \sum_{s \in S} \pi_s(p_m, q) D^*(q_s(p_m, q) || q) + o(\Delta_m),$$

where D^* is a divergence.

The divergence D^* is finite and convex in its first argument, and continuous in both arguments. $D^*(r || q)$ is twice-differentiable in r for r sufficiently close to q , with

$$\frac{\partial^2 D^*(r || q)}{\partial r^i \partial r^j} \Big|_{r=q} = \text{Diag}(q) k(q) \text{Diag}(q)$$

for some continuous matrix-valued function $k(q)$. For all q , $k(q)$ is positive semi-definite and symmetric, and satisfies $z^T k(q) z = 0$ for all $z \in \mathbb{R}^{|X|}$ that are constant in the support of q . If in addition the cost function satisfies Condition 5, then there exists a constant $m_g > 0$ such that

$$k(q) - m_g(\text{Diag}(q) - qq^T) \succeq 0.$$

Proof. See the appendix, section B.2. □

Going forward, we will use the notation

$$\bar{k}(q) = \text{Diag}(q) k(q) \text{Diag}(q)$$

to describe the Hessian of the divergence $D^*(r || q)$ evaluated at $r = q$, and observe that it is positive semi-definite.

Now that we have derived our approximation result, we turn to considering the continuous time limit of the dynamic rational inattention problem introduced in section §2.

4 Rational Inattention in Continuous Time

We next study the limit of the dynamic rational inattention problem described in section §2. We will derive these limits for several specific cases, depending whether the DM's has exponential discounting ($\beta < 1$ vs. $\beta = 1$) and on the nature of the flow information cost function $C(\cdot)$. In particular, we consider two cases for $C(\cdot)$: when there is a “preference for gradual learning” and when there is a “preference for discrete learning,” terms we define below. These two classes of flow cost functions will lead, respectively, to beliefs that move in small increments and beliefs that move in large increments.

Our most general convergence result proves that the discrete time dual problem converges $W(q_0, \lambda; \Delta)$ (defined in equation (11)) converges to our the continuous time dual value function, $W^+(q_t, \lambda)$, described in Definition 1. We also prove, for some value of λ^* , the convergence of the original problem $V(q_0; \Delta_n)$ to $W^+(q_t, \lambda^*)$.

Theorem 2. *Let Δ_m , $m \in \mathbb{N}$, denote a sequence such that $\lim_{m \rightarrow \infty} \Delta_m = 0$, let $V(q_0; \Delta_m)$ and $W(q_0, \lambda; \Delta_m)$ denote the discrete-time value functions defined in equations (10) and (11), and suppose that the flow cost function satisfies Conditions 1-5.*

The value function of the continuous time problem of Definition 1, $W(q_0, \lambda)$, is bounded, convex, and differentiable, and there exists a sub-sequence $n \in \mathbb{N}$ such that, for all λ if $\beta < 1$ and all $\lambda \in (0, \kappa c^{-\rho})$ if $\beta = 1$,

$$\lim_{n \rightarrow \infty} W(q_0, \lambda; \Delta_n) = W(q_0, \lambda).$$

There exists a λ^* such that

$$\lim_{n \rightarrow \infty} V(q_0; \Delta_n) = W(q_0, \lambda^*).$$

Moreover, it is without loss of generality to suppose that the diffusion terms (σ_s) of the optimal policy associated with $W(q_0, \lambda)$ are zero.

Proof. See the appendix, section B.6. □

Theorem 2 demonstrates the convergence of the original and dual problems, and shows (as part of the proof) that it is without loss of generality to assume there is no diffusion component. The intuition for the latter result, which we mentioned previously, is that it is possible to synthesize a “diffusion-like” process using the jump controls.

We next present the convergence result that is specific to the $\beta = 1$ case.

Corollary 1. Let Δ_m , $m \in \mathbb{N}$, denote a sequence such that $\lim_{m \rightarrow \infty} \Delta_m = 0$, let $V(q_0; \Delta_m)$ and $W(q_0, \lambda; \Delta_m)$ denote the discrete-time value functions defined in equations (10) and (11), and suppose that the flow cost function satisfies Conditions 1-5. If $\beta = 1$, then for all $\lambda \in (0, \kappa c^{-\rho})$, $\lim_{n \rightarrow \infty} W(q_0, \lambda; \Delta_n)$ is equal to the dual value function defined in Definition 2, and the limit $\lim_{n \rightarrow \infty} V(q_0; \Delta_n)$ is equal to the value function defined in Definition 2.

Proof. See the appendix, section B.10. □

These results demonstrate that the value functions in our discrete time problems converge to the value functions of our continuous time problem. However, the results leave unanswered two key questions: first, what are the dynamics of beliefs in the continuous time problem, and second, are these dynamics the limit of the optimal policies in the discrete time problem? To address these questions, we present conditions under which the beliefs will evolve gradually and conditions under which they will evolve in large increments.

4.1 Gradual Learning

We begin by defining what we call a “preference for gradual learning.” This condition describes the relative costs of learning via jumps in beliefs vs. continuously diffusing beliefs, which are governed by the properties of the divergence D^* that defines the cost function up to first order (Theorem 1).

Definition 3. The cost function $C(p, q; S)$ exhibits a “preference for gradual learning” if the associated divergence D^* and its Hessian $\bar{k}(q)$ satisfy, for all $q, q' \in \mathcal{P}(X)$ with $q' \ll q$,

$$D^*(q' || q) - (q' - q)^T \left(\int_0^1 (1-s) \bar{k}(sq' + (1-s)q) ds \right) (q' - q) \geq 0. \quad (16)$$

This preference is “strict” if the inequality is strict for all $q' \neq q$, and is “strong” if, for some $\delta > 0$ and some $m > 0$,

$$D^*(q' || q) - (q' - q)^T \left(\int_0^1 (1-s) \bar{k}(sq' + (1-s)q) ds \right) (q' - q) > m \|q' - q\|_X^{2+\delta}. \quad (17)$$

Recall that a divergence is a Bregman divergence if it can be written, using some convex function $H : \mathcal{P}(X) \rightarrow \mathbb{R}$, as

$$D_H(q' || q) = H(q') - H(q) - (q' - q)^T \cdot H_q(q).$$

To clarify the meaning of a preference for gradual learning, consider the “chain rule” (Cover and Thomas [2012]) that characterizes the Bregman divergences (including in particular the Kullback-Leibler divergence),

$$D_H(q' || q) + \sum_{s \in S} \pi_s D_H(q_s || q') = \sum_{s \in S} \pi_s D_H(q_s || q)$$

for all probability distributions π_s such that $\sum_{s \in S} \pi_s q_s = q'$. The following lemma demonstrates that, for any divergence D^* for which this chain rule is a “less-than-or-equal” inequality, there is a preference for gradual learning.

Lemma 2. *If the divergence D^* satisfies, for all $\pi_s \in \mathcal{P}(S)$ and $q, q', \{q_s\}_{s \in S} \in \mathcal{P}(X)$ such that $\sum_{s \in S} \pi_s q_s = q'$ and $q' \ll q$,*

$$D^*(q' || q) + \sum_{s \in S} \pi_s D^*(q_s || q') \leq \sum_{s \in S} \pi_s D^*(q_s || q),$$

and is twice-differentiable in its first argument, then the cost function $C(p, q; S)$ exhibits a preference for gradual learning.

Proof. See the appendix, section B.11. □

This chain-rule inequality has a simple interpretation— it is more costly to jump directly to the beliefs $\{q_s\}$ than to have beliefs travel first to q' and then on to $\{q_s\}$. It is straightforward to see why such an assumption leads directly to gradual learning, although it is worth noting that the preference for gradual learning is a weaker condition than this chain rule inequality. This chain rule inequality could also be called “super-additivity,” as the opposite of the “sub-additivity” assumption discussed by Zhong [2017].¹¹ In the next subsection, building on the results of Zhong [2019] and Zhong [2017], we will show in our model that that opposite inequality leads to immediate decision-making.

We begin by stating our gradual learning result for the $\beta = 1$ case. With a preference for gradual learning, we show that the value function converges to a continuous time problem with only a diffusion control (that is, without jumps). If the preference is strict, then the limiting process for beliefs must be a diffusion. If not (i.e. if D^* is a Bregman divergence, as in Zhong [2019]), there may be sequences of optimal policies in the discrete time models that converge to jump-diffusions, or even pure jump processes.

¹¹This is not the exact mirror of the sub-additivity assumption, because the inequality only applies to D^* , which describes the cost of a small amount of information, as opposed being an assumption on the cost function $C(\cdot)$ as in Zhong [2017].

Theorem 3. *Under the assumptions of Theorem 2, if $\beta = 1$ and the cost function satisfies a preference for gradual learning, there exists a sub-sequence, indexed by n , and a λ^* such that*

$$\lim_{n \rightarrow \infty} W(q_0, \lambda^*; \Delta_n) = \lim_{n \rightarrow \infty} V(q_0; \Delta_n) = W^+(q_0, \lambda^*) = V(q_0),$$

where

$$V(q_0) = \sup_{\{\sigma_t \in \mathbb{R}^{|\mathcal{X}| \times |\mathcal{X}|}, \tau \in \mathbb{R}_+\}} E_0[\hat{u}(q_\tau) - \kappa\tau]$$

subject to the constraints that $q_t^T \sigma_t = \vec{0}$,

$$dq_t = \text{Diag}(q_t) \sigma_t dB_t$$

and

$$\frac{1}{2} \text{tr}[\sigma_t \sigma_t^T k(q_t)] \leq \rho^{\frac{1}{\beta}} c.$$

If the cost function exhibits a strict preference for gradual learning, every convergent sub-sequence of belief processes $q_{t,n}^*$ associated with optimal policies in the discrete-time model converges in law to a diffusion.

Proof. See the appendix, section B.12. □

The sequence problem $V(q_0)$ has a straightforward interpretation— the DM controls the diffusion coefficient of her beliefs subject to a maximum rate of information acquisition defined by the information cost matrix function $k(q_t)$.

Under an additional assumption (described in the next section), a strong preference for gradual learning is not only sufficient but necessary for beliefs to follow a diffusion process in the $\beta = 1$ case. In particular, we will demonstrate that if, for all utility functions, the belief process in the continuous time limit is a diffusion, then the divergence D^* must exhibit a preference for gradual learning. However, to make this statement, we must be able to characterize the belief dynamics, which we are able to do given an additional assumption. We therefore postpone our proof of necessity to the next section.

We next turn to the case with discounting ($\beta < 1$). A remarkable result by Zhong [2019] (theorem 5 of that paper) demonstrates that, in a model very much like our continuous time limit, specialized to the case of only two states and with no linear time costs, the DM will generically choose to have no diffusion in her process, only jumps. This result can be understood in two parts. First, as discussed previously, it is without loss of generality to write the DM's belief process as a pure jump process, even when the law of the supremum over such processes is equivalent to the law of a diffusion.

The second part of the result of Zhong [2019] can be thought of as a (generic) lower bound on the magnitude of the jumps. That is, not only is it without loss of generality to consider a pure jump process, but the optimal policy of the DM is in fact a jump process with non-infinitesimal jumps. What is remarkable about this result, from the perspective of our results for the $\beta = 1$ case, is that it applies even when the cost function exhibits a preference for gradual learning. Zhong [2019] also shows, in the particular case of indifference to gradual learning (equality in equation (16)), which is

the setting for most of his results, that the beliefs jump all the way to stopping points, a result we will replicate below.

To understand how our $\beta = 1$ results are connected to the results of Zhong [2019], for the $\beta < 1$ case, we suppose that the cost function exhibits a “strong” preference for gradual learning, as defined in Definition 3 and equation (17). Under this assumption, we prove an upper bound on the size of the jumps, as a function of β , and show that in the limit as $\beta \rightarrow 1$, the bound converges to zero. In other words, there is no discontinuity— with discounting and a strong preference for gradual learning, the “pure jump process” for beliefs will become increasingly like a diffusion as the discount factor converges to unity. Recall that m and δ are part of the definition of a strong preference for gradual learning, and \bar{u} is the upper bound on the flow utility.

Theorem 4. *In problem defined in Definition 1, if the cost function satisfies a strong preference for gradual learning (equation (17)), then the optimal jump directions z_t^* satisfy*

$$\|z_t^*\|_X \leq \left(\frac{-\bar{u} \ln(\beta)}{m}\right)^{\delta^{-1}},$$

and $\lim_{\beta \rightarrow 1^-} \|z_t^*\|_X = 0$. Moreover, jumps always strictly increase the value function,

$$W(q_{t^-} + z_t^*, \lambda) > W(q_{t^-}, \lambda).$$

Proof. See the appendix, section B.13. □

The optimal policy for the discounting case features upward (in the sense of the value function) jumps and downward drift. The jumps become shorter and more frequent as the discount factor approaches one, eventually converging to a diffusion. The fact that jumps only increase and never decrease the value function is a consequence of the exponential discounting. Exponential discounting can be thought of as a penalty for delay that is increasing in the current level of the value function. For this reason, drifting upward and jumping downward are sub-optimal, because the former causes information to be acquired at a time when the cost of delay is high, and the latter acquires information at a time when the cost of delay is high rather than waiting for the cost of delay to decrease. We should note that the result that jumps always strictly increase the value function requires only a strict preference for gradual learning, rather than a strong preference, and such a result is without loss of generality (but not necessarily required) under a weak preference for gradual learning. Our result that jumps always increase the value function is reminiscent of a result in Che and Mierendorff [2019] and Zhong [2019], which is sometimes characterized as the DM seeking out confirmatory evidence. For reasons that we discuss below in our analysis of stopping times (and are illustrated by the one-sided learning example of Zhong [2019]), we prefer not to describe the result in this fashion.

Having analyzed the case of a preference for gradual learning, we next turn to the case of a preference for discrete learning,

4.2 Discrete Learning

Building on Zhong [2019] and Zhong [2017], to provide contrast to our results on discrete learning, we provide conditions under which, in the $\beta = 1$ case, the DM jumps immediately to stopping beliefs. We define what we call a “preference for discrete learning” if the divergence D^* satisfies the opposite of the “chain rule” inequality described previously (Lemma 2).

Definition 4. The cost function $C(p, q; S)$ exhibits a “*preference for discrete learning*” if divergence D^* associated with it satisfies, for all $\pi_s \in \mathcal{P}(S)$ and $q, q', \{q_s\}_{s \in S} \in \mathcal{P}(X)$ such that $\sum_{s \in S} \pi_s q_s = q'$ and $q' \ll q$,

$$D^*(q' || q) + \sum_{s \in S} \pi_s D^*(q_s || q') \geq \sum_{s \in S} \pi_s D^*(q_s || q). \quad (18)$$

The cost function $C(p, q; S)$ exhibits a “*strict preference for discrete learning*” if this inequality is strict for all $\pi_s, q, q', \{q_s\}_{s \in S}$ with $q' \neq q$ and $q_s \neq q'$ for some $s \in S$ such that $\pi_s > 0$.

If the cost function satisfies a preference for discrete learning, it is cheaper for the DM to jump to beliefs $\{q_s\}$ rather than visit the beliefs q' . Unsurprisingly, because this holds everywhere, it leads to optimal policies that stop immediately after jumping. Before discussing this result, we note that our definitions of a preference for discrete learning and a preference for gradual learning are not symmetric, although they are linked by Lemma 2. As mentioned previously, our definition of a preference for gradual learning includes cost functions that do not satisfy the chain rule inequality everywhere. The following lemma clarifies the relationship between a preference for gradual and discrete learning, showing that the strict versions of those preferences are incompatible.

Lemma 3. *Assume that the divergence D^* associated with the cost function $C(p, q; S)$ is twice-differentiable with respect to its first argument. If a cost function $C(p, q; S)$ exhibits a strict preference for gradual learning, it does not exhibit a preference for discrete learning. If a cost function $C(p, q; S)$ exhibits a strict preference for discrete learning, it does not exhibit a preference for gradual learning.*

Proof. See the appendix, section B.14. □

Note in particular that any twice-differentiable Bregman divergence (recall equation (8)) exhibits both a preference for gradual and discrete learning. Note also that there is no reason, in general, to expect either of the chain rule inequalities to hold over the entire simplex. That is, the cases we have analyzed are not exhaustive, and there may be cost functions that lead to large jumps in certain regions of the parameter space and small jumps or diffusions in other regions.

We now turn to characterizing the behavior of beliefs given a preference for discrete learning.

Theorem 5. *In the problem described by Corollary 1 (that is, $\beta = 1$), if the cost function satisfies a preference for discrete learning (Definition 4), then it is without loss of generality to suppose that, for all q_{t-} in the continuation region and any z_t^* that characterizes an optimal policy, $V(q_{t-} + z_t^*) = \hat{u}(q_{t-} + z_t^*)$. That is, without loss of generality, all jumps enter the stopping region.*

Proof. See the appendix, section B.15. □

The statement of Theorem 5 shows that is without loss of generality to assume that the DM stops immediately after a jump in beliefs. This result would be necessary, as opposed to without loss of generality, if the cost function exhibited a strict preference for discrete learning (although we will see below that such cost functions may not exist). In contrast, in the case of indifference (equation (18) holds with equality everywhere), both Theorem 3 and Theorem 5 hold. This happens if D^* is the KL divergence, or a Bregman divergence more generally. This observation implies that the two continuous time value functions must be identical, despite one being written as controlling a diffusion process and the other (without loss of generality) a pure jump process. We revisit this observation in the next section.

For completeness, we also provide an analog to our result about the magnitude and direction of jumps in the gradual learning case (Theorem 4) for the discounting case ($\beta < 1$). This result generalizes slightly a result in appendix A.3 of Zhong [2019]. With a preference for discrete learning, as with a preference for gradual learning, jumps will increase the value function. The intuition is essentially the same as the gradual learning case, and comes from the observation that with discounting, delay is particularly costly when the value function is high. However, unlike the gradual learning case, in which jumps are of bounded size, with a preference for discrete learning jumps are always immediately following by stopping.

Theorem 6. *In the problem defined in Definition 1, if $\beta < 1$ and the cost function satisfies a preference for discrete learning (Definition 4), then for all q_{t-} in the continuation region and any z_t^* that characterizes an optimal policy, $V(q_{t-} + z_t^*) = \hat{u}(q_{t-} + z_t^*)$. That is, without loss of generality, all jumps enter the stopping region. Moreover, jumps always strictly increase the value function,*

$$W(q_{t-} + z_t^*, \lambda) > W(q_{t-}, \lambda).$$

Proof. See the appendix, section B.16. □

Moving beyond the results of Zhong [2019], we provide the “only-if” result: if a divergence always results in large jumps and immediate stopping, then it must satisfy a preference for discrete learning. The intuition is that if it is always optimal to jump outside the continuation region, it cannot possibly be less costly, in the sense of the divergence D^* , to jump to some intermediate point. Otherwise, there would be some utility function for which such behavior is optimal.

Theorem 7. *In the problem defined in Definition 1, if there exists a cost function $C(p, q; S)$ and associated divergence D^* such that, for all strictly positive utility functions $u_{a,x}$, there exists an optimal policy such that $V(q_{t-} + z_t^*) = \hat{u}(q_{t-} + z_t^*)$ for all q_{t-} in the continuation region, then the cost function satisfies a preference for discrete learning.*

Proof. See the appendix, section B.17. □

We have proven the convergence of our discrete time problems to our continuous time rational inattention problems and shown that there are at least two different kinds of belief dynamics that might arise from the model, depending on whether there is a preference for gradual or discrete learning. We next provide an example of a family of cost functions that exhibit a preference for

gradual learning, and demonstrate that, subject to a regularity condition, Bregman divergences are the only cost functions that exhibit a preference for gradual learning.

4.3 Examples with a Preference for Discrete and Gradual Learning

Here, we provide an intuitive example of families of cost functions that generate preferences for gradual and discrete learning. We provide the examples in the context of posterior-separable cost functions,

$$C(p, q; S) = \sum_{s \in S} \pi_s(p, q) D^*(q_s(p, q) || q).$$

Note that our derivation of the model (Theorem 1) imposes a number of conditions on $D^*(p || q)$, including continuity and convexity in its first argument.

Cost functions exhibiting a (strict or strong) preference for gradual learning can be easily constructed from Bregman divergences. Suppose that

$$D^*(q' || q) = f(H(q') - H(q) - (q' - q)^T \cdot H_q(q)),$$

where $f(\cdot)$ is a twice-differentiable, strictly increasing, convex function with $f(0) = 0$, and $H(\cdot)$ is a strictly convex function on the simplex. It is straight-forward to observe that the Hessian of D^* evaluated at $q' = q$ is $f'(0)H_{qq}(q)$, and by convexity

$$D^*(q' || q) \geq f'(0)(H(q') - H(q) - (q' - q)^T \cdot H_q(q)),$$

implying that the divergence D^* satisfies a preference for gradual learning. This preference will be strict if $f(\cdot)$ is strictly convex, and will be strong if both $f(\cdot)$ and $H(\cdot)$ are strongly convex.

Constructing examples of cost functions with a strict preference for discrete learning has proven challenging. This class of cost functions may seem like a large class; however, as the following lemma demonstrates, subject to mild regularity conditions it only contains uniformly posterior separable cost functions (the Bregman divergence case).

Lemma 4. *Suppose the cost function $C(p, q; S)$ exhibits a preference for discrete learning and that the associated divergence D^* is continuously differentiable in both its arguments. Then D^* is a Bregman divergence and the cost function $C(p, q; S)$ is a uniformly posterior-separable cost function.*

Proof. See the appendix, section B.18. The proof builds on Banerjee et al. [2005]. □

As noted above, if D^* is a Bregman divergence, it exhibits a (non-strict) preference for discrete learning, and also a (non-strict) preference for gradual learning. Consequently, under mild regularity conditions, no cost functions exhibit a strict preference for discrete learning. In contrast, many cost functions exhibit a strict or strong preference for gradual learning, and many others fall into neither category (i.e. they have a preference for gradual learning in some parts of the parameter space and discrete learning in others). Combining this result with Theorem 7, we have demonstrated that, subject to mild regularity conditions, the jump-and-immediately-stop result of Zhong [2019] holds for all utility functions if and only if D^* is a Bregman divergence.

The results of this section naturally lead to the question of whether the different belief dynamics that occur under a preference for gradual or discrete learning lead to different predictions about the DM’s behavior. We explore this question in the next two sections.

5 The Equivalence of Static and Dynamic Models

In this section, we analyze the $\beta = 1$ continuous time model under a preference for gradual learning and a preference for discrete learning. The main result of this section is that, both with a preference for gradual learning and a preference for discrete learning (under an integrability assumption in the case of a preference for gradual learning, and assuming continuous differentiability in the case of a preference for discrete learning), the value function with $\beta = 1$ is equivalent to a static rational inattention problem with a uniformly posterior-separable cost function. Moreover, any twice-differentiable uniformly posterior-separable cost function can be justified through either of these routes.

This result has several implications. First, it demonstrates that both jump and diffusion-based models are tractable and that the value functions can be characterized without directly solving the associated partial differential equation. Second, it provides a micro-foundation for the uniformly posterior-separable cost functions that have been emphasized in the literature (see, e.g., Caplin et al. [2019]). Third, it proves that the two approaches are completely equivalent in terms of the predicted joint distribution of states $x \in X$ and actions $a \in A$. That is, any joint distribution of (x, a) that could be observed under discrete learning could be observed under gradual learning and vice versa.

One this last point, however, we do not mean to imply that the diffusion and jump processes are equivalent. Both of them endogenously will result in the same joint distribution of actions and states, but will have different predictions about the joint distribution of actions, states, and stopping times. As a consequence, considering stopping times can help differentiate the two models, and we take some steps toward this in the next section.

Finally, before presenting our results, we should emphasize that in the case of gradual learning, our results depend on additional assumptions beyond those implied by our derivation of the continuous time model. Our assumption for the gradual learning case is an integrability assumption, and will not hold generically. In contrast, in the discrete learning case, we make the mild additional assumption of continuous differentiability. Consequently, equivalence with static models holds for essentially all cost functions with a preference for discrete learning but only some cost functions for a preference for gradual learning, and essentially all cost functions with a preference for discrete learning generate the same joint distribution of actions and states as some cost function with a preference for gradual learning, but the reverse is not true.

5.1 Gradual Learning

To prove our equivalence result, we restrict our attention to information-cost matrix functions that are “integrable,” in the sense described by the following assumption.

Assumption 1. *There exists a twice-differentiable function $H : \mathbb{R}_+^{|X|} \rightarrow \mathbb{R}$ such that, for all q in the interior of the simplex,*

$$\text{Diag}(q)k(q)\text{Diag}(q) = H_{qq}(q), \quad (19)$$

where $H_{qq}(q)$ denotes the Hessian of H evaluated at q .

This class includes a number of information-cost matrix functions of interest: for example, it includes the case in which $k(q_t)$ is the inverse Fisher information matrix, which we will show corresponds to the standard rational inattention model, and the case in which $k(q_t)$ is the “neighborhood-based” function that we introduce in Hébert and Woodford [2018]. It is, however, a restrictive assumption. It rules out, for example, constant $k(q)$ (a hypothetical H would have asymmetric third-derivative cross-partials). We shall refer to the function H as the “entropy function,” for reasons that will become clear below. Note that $H(q)$ is convex, by the positive semi-definiteness of $k(q)$, and homogenous of degree one ($q^T \cdot H_{qq}(q) = \iota^T k(q)\text{Diag}(q)^{-1} = \vec{0}$).

For every convex function H , there is a Bregman divergence,

$$D_H(q_s||q) = H(q_s) - H(q) - (q_s - q)^T H_q(q),$$

and a corresponding uniformly posterior-separable cost function.

The problem we are analyzing is the result of Theorem 3, the problem with $\beta = 1$ and a diffusion process for beliefs. To analyze our continuous time problem, we begin by proving that the information constraint,

$$\frac{1}{2}\text{tr}[\sigma_t \sigma_t^T k(q_t)] \leq \rho^{\frac{1}{p}} c,$$

binds. Because the constraint binds, we can substitute the constraint into the HJB equation for the problem and obtain the following result:

Lemma 5. *At every point at which the value function $V(q_t)$ is twice-differentiable and the DM chooses not to stop, for all σ such that $q^T \sigma = \vec{0}$,*

$$\text{tr}[\sigma^T \{ \text{Diag}(q_t) V_{qq}(q_t) \text{Diag}(q_t) - \theta k(q_t) \} \sigma] \leq 0, \quad (20)$$

where $\theta = \frac{\kappa}{\rho^{\frac{1}{p}} c}$, with equality under the optimal policy.

Proof. See the appendix, Section B.19. □

The parameter θ , introduced in the lemma, describes the race between information acquisition and time in this model. The larger the penalty for delay, and the tighter the information constraint, the larger the parameter θ . We now describe our equivalence result, and then outline the key step of its proof, which relies on this lemma.

Theorem 8. *Under Assumption 1, the value function that solves the continuous time problem with $\beta = 1$ (Definition 2) and with a preference for gradual learning is*

$$V(q_0) = \max_{\pi \in \mathcal{P}(A), \{q_a \in \mathcal{P}(X)\}_{a \in A}} \sum_{a \in A} \pi(a) (u_a^T \cdot q_a) - \theta \sum_{a \in A} \pi(a) D_H(q_a || q_0),$$

subject to the constraint that $\sum_{a \in A} \pi(a)q_a = q_0$, where D_H is the Bregman divergence associated with the entropy function H that is defined by Assumption 1.

There exist maximizers π^* and q_a^* such that π^* is the unconditional probability, in the continuous time problem, of choosing a particular action, and q_a^* , for all a such that $\pi^*(a) > 0$, is the unique belief the DM will hold when stopping and choosing that action.

Proof. See the appendix, Section section B.20. □

The continuous time sequential evidence accumulation problem is equivalent to a static rational inattention problem, with a particular uniformly posterior-separable cost function,

$$C(p, q_0; S) = \sum_{s \in S} \pi_s(p, q_0) D_H(q_s(p, q_0) || q_0), \quad (21)$$

and with the signal space S identified with the set of possible actions A .

The mutual information cost function (6) proposed by Sims is one such cost function. In this case, the entropy function H is the negative of Shannon’s entropy (4), the corresponding information-cost matrix function is the inverse Fisher information matrix $k(q) = \text{Diag}(q) - qq^T$, the Bregman divergence is the Kullback-Leibler divergence (5), and the information measure defined by (21) is mutual information (6). Thus Theorem 8 provides a foundation for the standard static rational inattention model, and hence for the same predictions regarding stochastic choice as are obtained by Matějka et al. [2015]. On the other hand, Theorem 8 also implies that other cost functions can also be justified. Indeed, any (twice-differentiable) uniformly posterior-separable cost function (21) can be given such a justification, by choosing the information cost matrix function defined by equation (19).

However, not all information cost matrix functions are reasonable. We interpret the information cost matrix function $k(q)$ as a description of how hard it is to distinguish any pair of states. In many economic applications, there is a natural ordering or structure of the states, and we would like the information cost matrix function and the associated entropy function and Bregman divergence to reflect this structure. In Hébert and Woodford [2018], we propose such a cost function.

We next outline the key step of our proof. Our proof strategy is best described as “guess and verify,” in that we start with the static value function described in Theorem 8 and then show that it is the value function of the continuous time model described in Theorem 3. The key step of the proof is to show that the static value function satisfies (20) in Lemma 5.¹² For expositional purposes, we will assume that the optimal policies of the static model, $\pi^*(a; q_0)$ and $q_a^*(q_0)$, are differentiable with respect to q_0 and strictly interior (we do not require these assumptions in the proof).

We begin by examining the first-order conditions with respect to q_a , and applying the envelope theorem. Let $\kappa(q_0)$ denote the vector of multipliers on the constraint that $\sum_{a \in A} \pi(a)q_a = q_0$. We

¹²Technical footnote: the “anywhere the value function is twice differentiable” caveat of Lemma 5 is relevant for our problem. The PDE described by equation (20) is “degenerate elliptic” and hence will not in general have a classical solution. Indeed, we do not prove that our static value function is twice differentiable everywhere, and suspect it is not at points where the “consideration set” (the set of actions with $\pi(a) > 0$, Caplin et al. [2018]) changes. In our proof, we establish that the static problem value function is convex and continuously differentiable, which is sufficient to invoke a generalized version of Ito’s lemma for convex functions to verify that the static value function is the solution to the continuous time problem.

have the first-order condition (FOC) and envelope theorem condition (ET),

$$\text{FOC: } u_a - \kappa(q_0) - \theta H_q(q_a^*(q_0)) + \theta H_q(q_0) = 0, \forall a \in A, \quad (22)$$

$$\text{ET: } V_q(q_0) = \kappa(q_0) + \sum_{a \in A} \pi^*(a; q_0) (q_a^*(q_0) - q_0)^T \cdot H_{qq}(q_0) = \kappa(q_0).$$

Now consider a perturbation $q_0 \rightarrow q_0 + \epsilon z$, for some tangent vector z . Combining the FOC and ET, and then differentiating with respect to ϵ and evaluating at $\epsilon = 0$,

$$V_{qq}(q_0) \cdot z = \theta H_{qq}(q_0) \cdot z - \theta H_{qq}(q_a^*(q_0)) \cdot \frac{dq_a^*(q_0 + \epsilon z)}{d\epsilon} \Big|_{\epsilon=0}, \forall a \in A. \quad (23)$$

Observe that, due to the constraint, $\sum_{a \in A} \frac{d(\pi^*(a; q_0 + \epsilon z) q_a^*(q_0 + \epsilon z))}{d\epsilon} \Big|_{\epsilon=0} = z$. Multiplying both sides of equation (23) by $\frac{d(\pi^*(a; q_0 + \epsilon z) q_a^*(q_0 + \epsilon z)^T)}{d\epsilon} \Big|_{\epsilon=0}$, and then taking sums,

$$\begin{aligned} z^T \cdot V_{qq}(q_0) \cdot z &= \theta z^T \cdot H_{qq}(q_0) \cdot z \\ &\quad - \theta \sum_{a \in A} \pi^*(a; q_0) \left(\frac{dq_a^*(q_0 + \epsilon z)}{d\epsilon} \Big|_{\epsilon=0} \right)^T \cdot H_{qq}(q_a^*(q_0)) \cdot \frac{dq_a^*(q_0 + \epsilon z)}{d\epsilon} \Big|_{\epsilon=0} \\ &\quad - \theta \sum_{a \in A} \frac{d\pi^*(a; q_0 + \epsilon z)}{d\epsilon} \Big|_{\epsilon=0} q_a^*(q_0)^T \cdot H_{qq}(q_a^*(q_0)) \cdot \frac{dq_a^*(q_0 + \epsilon z)}{d\epsilon} \Big|_{\epsilon=0}. \end{aligned}$$

By Assumption 1, $q^T \cdot H_{qq}(q) = 0$, and hence the last line in this expression is zero. By the convexity of H , the summation on the second line is positive. Therefore, by Assumption 1,

$$z^T \cdot V_{qq}(q_0) \cdot z \leq z^T \cdot \text{Diag}(q_0)^{-1} k(q_0) \text{Diag}(q_0)^{-1} \cdot z,$$

establishing that (20) holds. To show that there is a direction z^* in which (20) holds with equality, it is sufficient to show that $\frac{dq_a^*(q_0 + \epsilon z^*)}{d\epsilon} \Big|_{\epsilon=0} = 0$ for all $a \in A$. In any direction z spanned by the initial $q_a^*(q_0) - q_0$, it is not optimal for the DM to change the $q_a^*(q_0)$, only the probabilities $\pi^*(a, q_0)$ (this property, ‘‘Locally Invariant Posteriors,’’ was shown by Caplin et al. [2019]). Thus, any of these directions can serve as z^* .¹³

We conclude that all continuous time models with gradual learning that also satisfy our integrability condition are equivalent to a static model with a uniformly posterior-separable cost function, and that any such static model can be justified from some model with gradual learning. We next show that the same set of static models can be justified from a model with discrete learning. Before proceeding, however, we observe that this result allows us to demonstrate that a preference for gradual learning is necessary for beliefs to always result in a diffusion process, provided that Assumption 1 holds.

Corollary 2. *In the problem defined in Definition 2 (the $\beta = 1$ case), if a cost function $C(p, q; S)$, associated divergence D^* , and information cost matrix function $k(q)$ satisfying Assumption 1 are*

¹³That any such direction can serve as z^* indicates that there are (usually) many optimal policies in the continuous time problem that achieve the same value function. Intuitively, at each point, if the DM does not learn in some particular direction, she could always learn in that direction in the next instant.

such that, for all strictly positive utility functions $u_{a,x}$, there exists an optimal policy such that $\bar{\psi}_s = 0$ everywhere in the continuation region (meaning beliefs follow a diffusion process), then the cost function satisfies a preference for gradual learning.

Proof. See the appendix, section B.21. □

5.2 Discrete Learning

The result with a preference for discrete is an immediate corollary of Lemma 4 and the preceding Theorem 8 (the result with gradual learning).

Corollary 3. *Assume the cost function $C(p, q; S)$ exhibits a preference for discrete learning, and the associated divergence D^* is continuously differentiable. Then D^* is a Bregman divergence and value function that solves the continuous time problem with $\beta = 1$ (Definition 2) is identical to the static rational inattention problem described in Theorem 8 with D^* in the place of D_H .*

Proof. Immediate from Lemma 4 and Theorem 8. □

The requirement that D^* be continuously differentiable is likely unnecessary for the result. Our derivation of D^* requires that it be convex in its first argument and continuous in both arguments. Applying “mollification” techniques for convex functions (see for example Banerjee et al. [2005]) would likely allow us to extend the result to all D^* functions satisfying the conditions of Theorem 1.

Given any uniformly posterior-separable cost function in a static rational inattention model, by setting D^* equal to the Bregman divergence associated with that cost function, we can justify that static model as the result of a dynamic model with a preference for discrete learning. We therefore conclude that models with a preference for gradual learning satisfying our integrability condition and models with a preference for discrete learning are indistinguishable from the perspective of their predictions about the joint distribution of states and actions.¹⁴ In the next section, we begin to explore how information about stopping times can be used to distinguish the models.

6 Implications for Response Times

Because our model is dynamic, the observable behavior of the DM includes not only the joint distribution of actions and states, but also information about the length of time taken to decide, including how this may vary depending on the action and the state. The psychophysics literature gives considerable emphasis to facts about response times as a window on the nature of decision processes (e.g., Ratcliff and Rouder [1998]). In economic contexts as well, response times provide important information that can be used to discriminate between models, even when response times themselves are not what the economic analyst cares about. For example, Clithero [2018] and Alos-Ferrer et al. [2018] argue that preferences can more accurately be recovered from stochastic choice data when data on response times are used alongside observed choice frequencies.

¹⁴In situations in which the static rational inattention problem does not itself have a unique solution, we have not ruled out the possibility that the models with discrete and gradual learning will make different predictions. However, we have no reason to believe this is the case.

Here we propose that data on response times can be used to discriminate between alternative information-cost specifications. We will show that cost functions that are equivalent in the sense of implying the same value function nevertheless make different predictions about the stopping time conditional on taking a particular action. Consequently, data on stopping times can inform us about whether there is a preference for gradual learning or for learning through discrete jumps. Interestingly, it is possible to distinguish between these two hypotheses even when (as in the problems considered here) actions are taken only infrequently.¹⁵

We illustrate how response time data can discriminate between the two hypotheses using a simple example. In this example, we consider the no-discounting limit, $\beta \rightarrow 1$. We focus on this limit for several reasons. First, with regards to decision-making experiments, the cumulative discount factor over the length of time required to make a decision (often seconds or minutes) should be small. Second, it is easier to characterize stopping times when beliefs are a diffusion process than when they are a sequence of small jumps. Third, as noted by Fudenberg et al. [2018], behavior in the $\beta < 1$ case is not invariant to transformations of the utility function. Fourth, and most significantly, our results from the previous section demonstrate that in this case the gradual-learning and jump-learning approaches are equivalent in terms of their predictions for the joint distribution of actions and states. We consider the limit $\beta \rightarrow 1$, rather than simply assuming no discounting, as this allows us to choose, in the case of a preference for discrete learning, between the multiple optimal policies for the $\beta = 1$ case.

We also restrict attention to the case of only two states. This allows us to ignore the integrability condition in the gradual learning case, as it is always satisfied. We can then compare cost functions with a preference for gradual and discrete learning that generate the exact same predictions for the joint distribution of actions and signals. Second, with $\beta = 1$, two states, and a preference for gradual learning, the optimal policies that generate the stochastic process for beliefs are uniquely determined by the results of Theorem 3. Similarly, in the limit as β approaches 1 from below, with two states and a preference for discrete learning, the optimal policies that generate the stochastic process for beliefs are uniquely determined by the results of Theorem 6.¹⁶

The difference that we illustrate concerns the way in which changes in the payoffs associated with particular actions is predicted to affect the stopping time dynamics. We will show that a preference for gradual learning implies an invariance result, in which a shift in the level of the utility for all actions in a given state does not change the joint distribution of actions, states, and stopping times. In contrast, we will show that such a shift can radically alter the distribution of stopping times under a preference for discrete learning.

Label the states $X = \{G, B\}$, let the two actions available be $A = \{L, R\}$, and suppose that the optimal action in state G is L , while the optimal action in state B is R . We assume that these actions are strictly optimal, meaning that $u(L, G) > u(R, G)$ and $u(R, B) > u(L, B)$ (without these assumptions, the DM should immediately stop and not gather any information). We also assume,

¹⁵It would obviously be easier to tell whether beliefs evolve continuously or in discrete jumps in a case where the DM is required to continuously adjust some response variable that can provide an indicator of her current state of belief.

¹⁶In the $\beta = 1$ case, with a preference for discrete learning, there are multiple optimal policies. However, only one of these is the limit of the optimal policies from the $\beta < 1$ case.

without loss of generality, that taking the optimal action in state G generates a weakly higher payoff than taking the optimal action in state B , $u(L, G) \geq u(R, B)$.

With two states, we can interpret the beliefs q as a scalar, and we adopt the convention that q is the probability of state G . We compare two different situations, one in which the value function $W^+(q, \lambda)$ is “u-shaped” and one in which the value function $W^+(q, \lambda)$ is upward-sloping. The “u-shaped” value function occurs when $u(R, B) > u(R, G)$; that is, when taking the correct action in state B generates a higher payoff than taking the wrong action in state G . The downward-sloping value function occurs when $u(R, B) \leq u(R, G)$. In the case, the state G is simply better for the DM than the state B , regardless of the action the DM takes. For both of these models, there will be unique beliefs $0 < q_R < q_L < 1$ associated with taking actions R and L , respectively.

The objects we will study are the cumulative probabilities of stopping by time t and taking action L or R , $F_L(t)$ and $F_R(t)$. We also study the conditional (on states G and B) versions of these processes, $F_{L,G}(t), F_{L,B}(t), F_{R,G}(t), F_{R,B}(t)$. We construct two different numerical examples, described in terms of the utility function.

$$\text{Case 1. } u(L, G) = u(R, B) = 3, u(L, B) = u(R, G) = 2$$

$$\text{Case 2. } u(L, G) = 3, u(R, B) = u(R, G) = 2, u(L, B) = 1$$

Note that in both cases, utility increases by one when making the correct decision relative to the wrong decision ($u(L, G) - u(R, G) = u(R, B) - u(L, B) = 1$). What differentiates the two cases is that in case 2 utilities are lower by one in state B . This is convenient because it implies that the stopping beliefs q_R and q_L are identical in the two cases. This result in fact applies regardless of the number of states and actions, as the following lemma describes.

Lemma 6. *In the problem described by Definition 2 (that is, $\beta = 1$), if $u_{a,x}(\epsilon) = u_{a,x} + \epsilon v_x$ for some $v \in \mathbb{R}^{|X|}$, then the set of joint distributions of stopping times, actions, and states under optimal policies is identical for all $\epsilon \in \mathbb{R}$.*

Proof. This follows immediately from the invariance of the value function to level shifts in the utility function. \square

This conclusion depends only on the assumption that $\beta = 1$. It also provides a way of testing if $\beta = 1$ or not— does shifting all rewards up or down change the observed distribution of stopping times, actions, and states? Note also that this result holds regardless of whether the cost function exhibits a preference for gradual or discrete learning. However, the import of the result is different in the two cases.

With a strict preference for gradual learning, in the two-state case, there is a unique optimal policy involving a diffusion, which generates a unique joint distribution of stopping times, actions, and states, and this unique distribution will occur with both the case 1 and case 2 utility functions. Moreover, this unique optimal policy is the limit of the optimal policies in the $\beta < 1$ case. This result, for the case of a strict preference for gradual learning, does depend on the assumption that $\beta = 1$. That is, with $\beta < 1$, different policies will be optimal under case 1 and case 2 utility functions,

although they will remain qualitatively similar and converge to an identical limit as β approaches one.

In contrast, with a preference for discrete learning, there are many optimal policies, all of which lead to the same joint distribution of actions and states, but which vary in terms of their joint distribution of actions, states, and stopping times. Moreover, in the two state case, only one policy from this set is the limit of the optimal policies from the $\beta < 1$ case (which are themselves unique), and which policy corresponds to this limit depends on whether the utility function is from case 1 or case 2.

Putting together Theorem 8, Corollary 3, and Lemma 6, we proceed by choosing a cost function that satisfies a preference for discrete learning, and construct the cost function with a strict preference for gradual learning that makes equivalent predictions about the joint distributions of actions and states. We then analyze the stopping time distributions $F_L(t)$ and $F_R(t)$ with gradual learning and for both cases with a preference for discrete learning.

Because it is a standard in the literature, we choose mutual information as our cost function that satisfies a preference for discrete learning. The associated divergence D^* is the Kullback-Leibler divergence, which is a Bregman divergence. To construct another cost function with a strict preference for gradual learning that makes identical predictions, we use a posterior-separable cost function with

$$D^*(p||q) = f(D_{KL}(p||q)), \quad (24)$$

where f is any strictly-increasing, strictly convex function with $f(0) = 0$ and $f'(0) = 1$ (all such functions will generate the same predicted behavior).

To complete the model specification, we set $\kappa = \rho^{\frac{1}{\rho}}c = 1$, and assume a prior $q_0 = \frac{3}{5}$. Using Theorem 8 and Corollary 3, it follows that the stopping boundaries q_L and q_R solve

$$\begin{aligned} \max_{\pi_L \in [0,1], q_L \in [0,1], q_R \in [0,1]} & \pi_L q_L u(L, G) + \pi_L (1 - q_L) u(L, B) + \\ & (1 - \pi_L) q_R u(R, G) + (1 - \pi_L) (1 - q_R) u(R, B) - \\ & \pi_L D_{KL}(q_L || q_0) - (1 - \pi_L) D_{KL}(q_R || q_0), \end{aligned}$$

subject to $\pi_L q_L + (1 - \pi_L) q_R = \frac{3}{5}$. The unique solution to this problem, for both case 1 and case 2, is $q_L = \frac{e}{1+e} \approx 0.73$ and $q_R = \frac{1}{1+e} \approx 0.27$. As mentioned previously, these stopping boundaries apply regardless of whether the cost function exhibits a preference for discrete learning or a strict preference for gradual learning.

We begin by discussing the case of a strict preference for gradual learning. This case is in one sense simple. Beliefs follow a diffusion process, eventually hitting one of two boundaries. Consequently, the cumulative distributions $F_L(t)$ and $F_R(t)$ are the first-passage times of a diffusion process. The diffusion coefficient of this process is determined by the information cost matrix function $k(q_t)$ (see Theorem 3). The resulting CDFs $F_L(t)$ and $F_R(t)$ will generally have ‘‘sigmoid’’ shapes. What makes this case complicated is that the diffusion coefficient is usually not constant, and as a result we cannot rely on analytical results about normal distributions.

Fortunately, in our example, the resulting diffusion process is well-studied. When D^* is defined

as in equation (24), the associated Hessian matrix \bar{k} is the Fisher information matrix. As a result, the dynamics of beliefs follow a diffusion that is studied in the mathematical genetics literature as the ‘‘Fisher-Wright’’ model.¹⁷ The lemma below summarizes these results.

Lemma 7. *In the problem described by Corollary 1 (that is, $\beta = 1$), if the divergence D^* is defined by equation (24), the process for beliefs in the two-state model is a diffusion,*

$$dq_t = (2\rho^{\frac{1}{\rho}} c q_t (1 - q_t))^{\frac{1}{2}} dB_t.$$

Conditional on the true state being G , the process is

$$dq_t = 2\rho^{\frac{1}{\rho}} c (1 - q_t) dt + (2\rho^{\frac{1}{\rho}} c q_t (1 - q_t))^{\frac{1}{2}} dB_t,$$

and conditional on the true state being B the process is

$$dq_t = -2\rho^{\frac{1}{\rho}} c q_t dt + (2\rho^{\frac{1}{\rho}} c q_t (1 - q_t))^{\frac{1}{2}} dB_t.$$

Proof. See the appendix, section B.22. □

Armed with this diffusion process, we can solve for the functions $F_L(t)$ and $F_R(t)$. Let $\phi^L(q, t, \tau)$ be the probability of hitting q_L before time τ given $q_t = q$, and observe that $F_L(t) = \phi^L(q_0, 0, t)$. By the Markov property, $\phi^L(q, t, \tau) = \phi^L(q, 0, \tau - t)$, which we abbreviate as $\phi^L(q, s)$.

By the usual dynamic programming arguments, we have

$$\phi_s^L(q, s) = \rho^{\frac{1}{\rho}} c q (1 - q) \phi_{qq}^L(q, s),$$

with the boundary conditions $\phi^L(q_L, s) = 1$ for all $s \geq 0$, $\phi(q_R, s) = 0$ for all $s \geq 0$, and $\phi(q, 0) = 0$ for all $q \in (q_L, q_R)$. The same partial differential equation, but with different boundary conditions, characterizes $F_R(t)$. The conditional (on G or B) partial differential equations involve a first-order term for the drift, but are otherwise identical.

We numerically solve this PDE, and show in figure 1 that (intuitively) both $F_L(t)$ and $F_R(t)$ have the expected sigmoid shapes. Note, as discussed above, that these stopping time distributions apply in both case 1 and case 2. We show conditional (on G or B) CDFs in figure 2.

¹⁷The connection between these models, as indicated by R. A. Fisher’s name appearing in both the Fisher-Wright model and the Fisher information matrix, is not a coincidence. One minor difference between our application and the Fisher-Wright model is the location of the boundaries. Morris and Strack [2019] study this process as a special case of our framework.

Figure 1: Stopping Times, Strict Preference for Gradual Learning

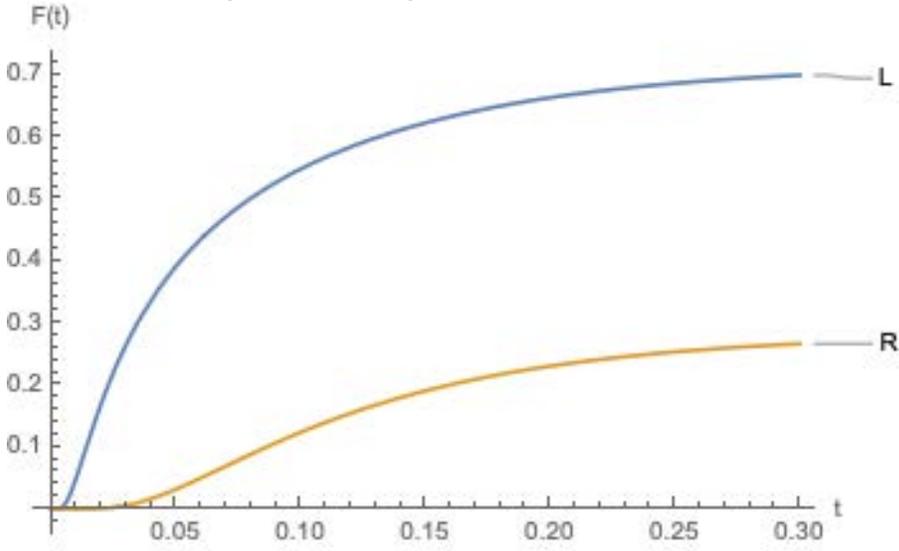
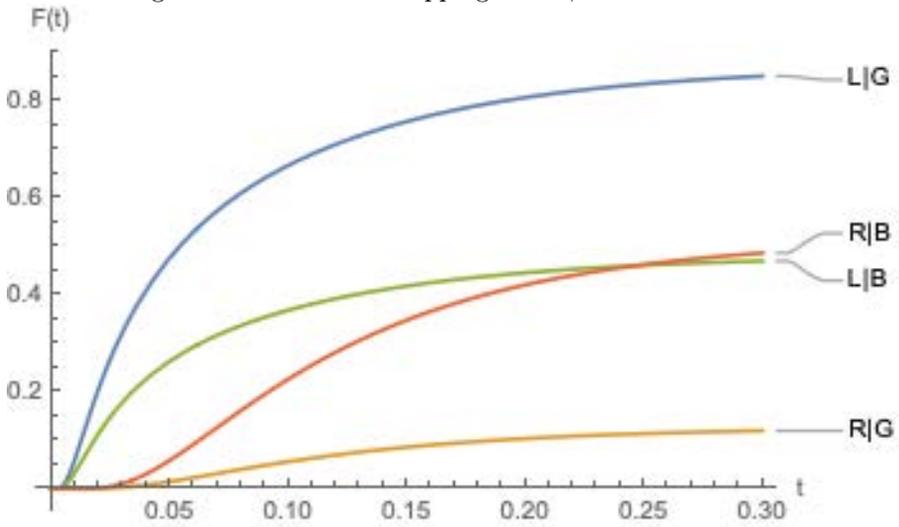


Figure 2: Conditional Stopping Times, Strict Preference for Gradual Learning



We now turn to the case of discrete learning. Consider first the symmetric case, case 1. In this case, because the value function is symmetric, it is minimized at $q = \frac{1}{2}$. It follows by our result that the value function always jumps up and drifts down (or the equivalent result in Zhong [2019]) that while $q > \frac{1}{2}$, the intensity of jumping to q_L is

$$\bar{\psi}_L(q) = \frac{\rho^{\frac{1}{2}} c}{D_{KL}(q_L||q)},$$

and the intensity of jumping to q_R is zero. After this point, this hazard rates will both be equal to half this value. Define $\tau_{\frac{1}{2}}$ as the time at which q_t reaches $\frac{1}{2}$. Because beliefs are martingales, the likelihood of this occurring (that is, of failing to jump before reaching $q = \frac{1}{2}$) is $\frac{q_L - q_0}{q_L - \frac{1}{2}}$. For all $t > \tau_{\frac{1}{2}}$,

$$\begin{aligned} F_L(t) &= \left(1 - \frac{q_L - q_0}{q_L - \frac{1}{2}}\right) + \frac{1}{2} \left(\frac{q_L - q_0}{q_L - \frac{1}{2}}\right) (1 - \exp(-\bar{\psi}_L(\frac{1}{2})(t - \tau_{\frac{1}{2}})), \\ F_R(t) &= \frac{1}{2} \left(\frac{q_L - q_0}{q_L - \frac{1}{2}}\right) (1 - \exp(-\bar{\psi}_L(\frac{1}{2})(t - \tau_{\frac{1}{2}})). \end{aligned}$$

Prior to time $\tau_{\frac{1}{2}}$, conditional on not stopping, the beliefs $q(t)$ drift towards $q = \frac{1}{2}$,

$$q'(t) = \bar{\psi}_L(q(t))(q(t) - q_L),$$

and the probability of stopping accumulates at a hazard rate based on $\bar{\psi}_L(q(t))$,

$$F'_L(t) = (1 - F_L(t))\bar{\psi}_L(q(t)).$$

Solving this system of differential equations with initial conditions $F_L(0) = 0$ and $q(0) = q_0$, and finding the unique time value where $q(\tau_{\frac{1}{2}}) = \frac{1}{2}$ and $F_L(\tau_{\frac{1}{2}}) = \frac{q_0 - \frac{1}{2}}{q_L - \frac{1}{2}}$, which occur simultaneously, characterizes the system.

Our numerical solution is shown below in figure 3, and the conditional stopping times are shown in figure 4. Note that, aside from a few kinks and the like, the stopping time distribution is almost identical to the case of a preference for gradual learning.

Figure 3: Stopping Times, Preference for Discrete Learning, Case 1

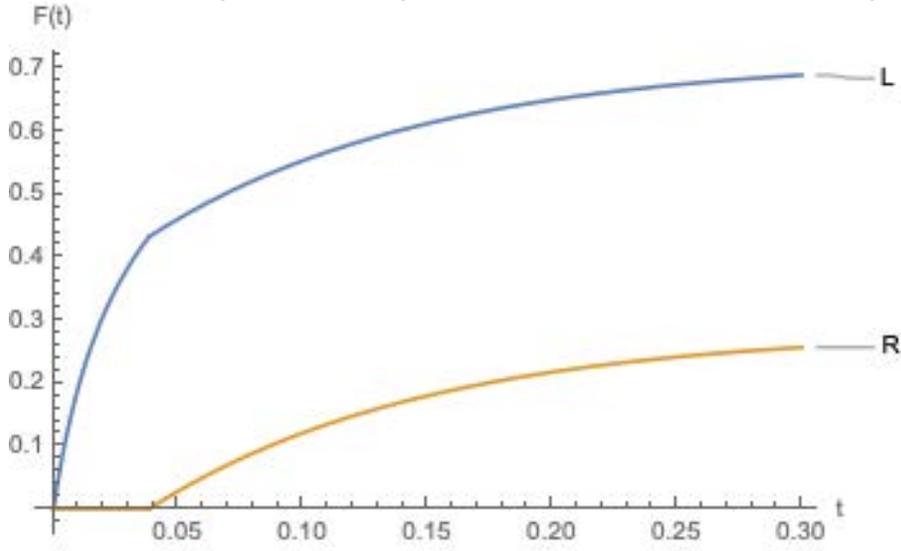
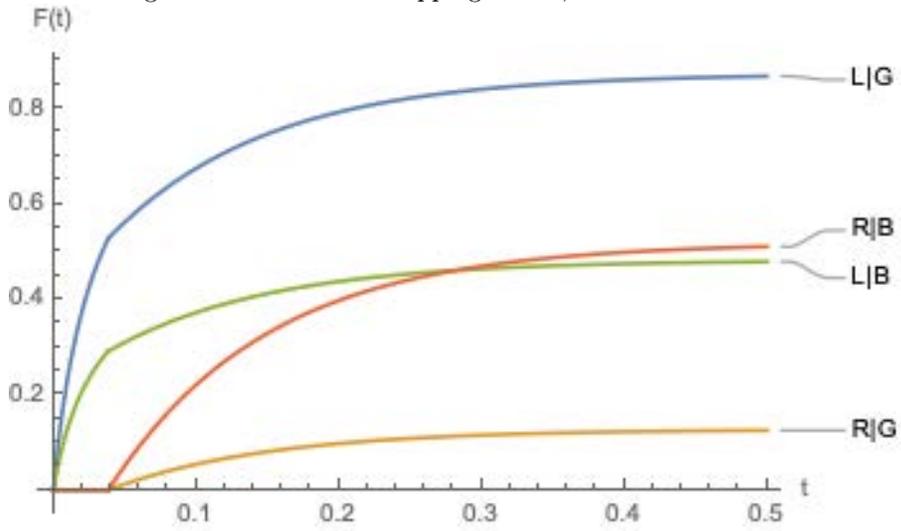


Figure 4: Conditional Stopping Times, Preference for Discrete Learning, Case 1



Now consider the asymmetric case, case 2. In this case, the value function is minimized at q_R , due to smooth pasting. That is, $\frac{d}{dq}\hat{u}(q)|_{q=q_R} = 0$ because $u(R, B) = u(R, G)$, and by the convexity of the value function¹⁸, it follows that the value function is increasing on its domain $q \in [q_R, q_L]$. It immediately follows by Theorem 6 that, in the $\beta < 1$ case, the DM will always attempt to jump to q_L and drift towards q_R , a result Zhong [2019] calls one-sided learning.

The system that characterizes beliefs is identical to the $t < \tau_{\frac{1}{2}}$ domain of the previous case, stopping at $q = q_R$ and $F_L(\tau_R) = \frac{q_0 - q_R}{q_L - q_R}$ instead. After τ_R , the DM immediately decides to stop and choose action R . As a result, the cumulative distributions of stopping times look quite a bit different than the corresponding functions from case 1. We show these results below, in figure 5, and the conditional results in figure 6.

We are not aware of any direct experimental evidence that manipulates the level of payoffs while preserving the incentive for correct choices, though we find the predicted abrupt effect of a small change in payoffs implausible.¹⁹ Note that the abrupt change in behavior under a preference for discrete learning depends on use of the $\beta \rightarrow 1$ limit as an optimal policy selection device. In a model with $\beta < 1$, the DM's behavior is not invariant to transformations of the utility function. The use of $\beta \rightarrow 1$ as a selection device yields predicted behavior even in the limit of no discounting that continues not to satisfy the invariance property.²⁰

¹⁸This follows from appendix Lemma 13, which shows convexity for the discrete value functions, and Theorem 2.

¹⁹An experiment to test this prediction ought to be feasible.

²⁰In our view, this issue should cast doubt on the usefulness of theories of behavior in the undiscounted case that are premised on the importance of continuity with behavior in the discounted case.

Figure 5: Stopping Times, Preference for Discrete Learning, Case 2

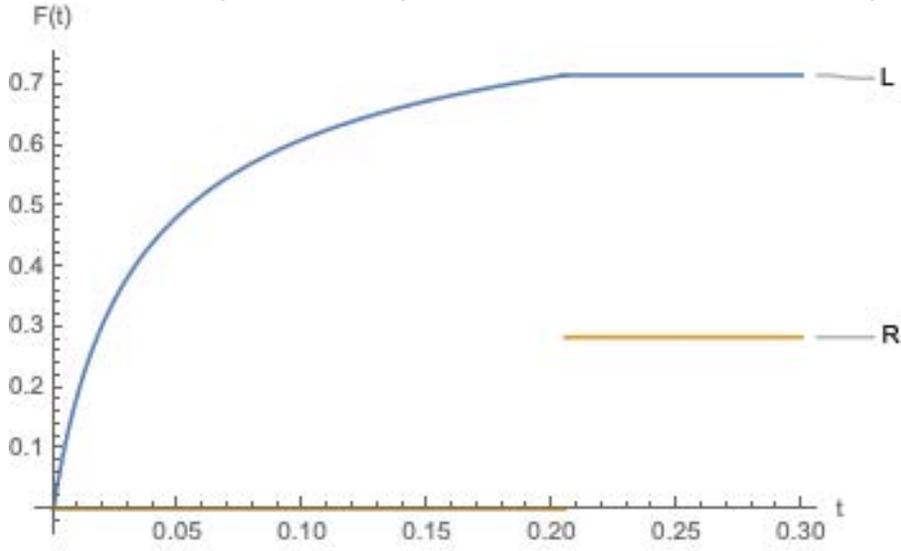
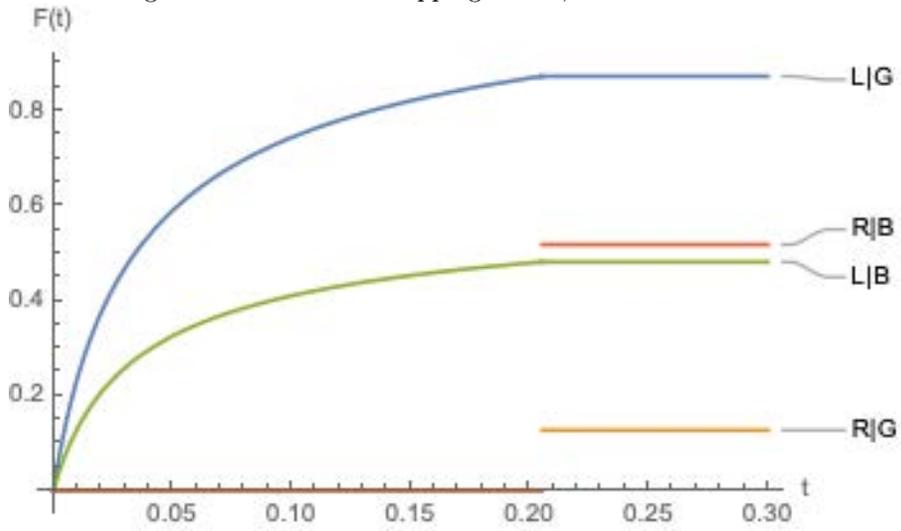


Figure 6: Conditional Stopping Times, Preference for Discrete Learning, Case 2



7 Discussion and Conclusion

We have derived a continuous-time rational-inattention model as the limit of a discrete-time sequential evidence accumulation problem. In the limit of a very large number of successive signals, each of which is only minimally informative, only the local properties of the flow cost function matter. Using these properties, we have demonstrated (without discounting) cases in which beliefs converge to either a pure diffusion process or a pure jump process. With discounting, we have described a more general limiting problem, and shown that it may result in “diffusion-like” pure jump processes. Furthermore, in the case without discounting, we have demonstrated that the resulting behavior (under an additional integrability assumption in the gradual learning case) is equivalent to the behavior predicted by a static rational inattention model. This equivalence provides a relatively simple way of generating predictions about the effects of variation in the opportunity cost of time for the probability of choosing different actions, without having to numerically solve a complex dynamic model.

We have left unresolved the question of whether it is appropriate to use the discounting or no-discounting models, and whether a preference for gradual learning is a reasonable way to characterize the evolution of beliefs. It is worth noting that in many decision-making contexts, the time it takes to make a decision (seconds or minutes) is likely to call for a cumulative discount factor (β^τ) very close to one. As a result, the limiting case of $\beta = 1$ may provide a useful and tractable approximation. We also note that when our key divergence D^* is a Bregman divergence, both our diffusion and jump results apply. However, this does not imply that everything is the same about the two problems—in particular, the joint distribution of decision times and actions taken may differ between the two models. This joint distribution is the subject of a great deal of existing research (e.g. Fudenberg et al. [2018]) and results in this literature might help determine whether a preference for gradual or discrete learning provides a better description of decision making.

The continuous time limit we have derived in this paper can serve as a foundation for analytically tractable models of rational inattention. One strength of our approach is its generality—we have imposed relatively minimal assumptions on the nature of information costs, and yet derived models that are specific and tractable enough to be tested in data. Exploring the properties of these models, and in particular whether jump processes or diffusions better characterize beliefs, and what cost functions should be employed, is the next step in this research agenda.

References

- Syed Mumtaz Ali and Samuel D Silvey. A general class of coefficients of divergence of one distribution from another. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 131–142, 1966.
- Carlos Alos-Ferrer, Ernst Fehr, and Nick Netzer. Time will tell: Recovering preferences when choices are noisy. *Unpublished manuscript*, October 2018.

- Shun-ichi Amari and Hiroshi Nagaoka. *Methods of information geometry*, volume 191. American Mathematical Soc., 2007.
- Nihat Ay, Jürgen Jost, Hông Vân Lê, and Lorenz Schwachhöfer. Information geometry and sufficient statistics. *Probability Theory and Related Fields*, pages 1–38, 2014.
- Arindam Banerjee, Xin Guo, and Hui Wang. On the optimality of conditional expectation as a Bregman predictor. *IEEE Transactions on Information Theory*, 51(7):2664–2669, 2005.
- David Blackwell. Equivalent comparisons of experiments. *The annals of mathematical statistics*, 24(2):265–272, 1953.
- Jerome R Busemeyer and James T Townsend. Decision field theory: A dynamic-cognitive approach to decision making in an uncertain environment. *Psychological Review*, 100:432–459, 1993.
- Andrew Caplin and Mark Dean. Revealed preference, rational inattention, and costly information acquisition. *American Economic Review*, 105(7):2183–2203, 2015.
- Andrew Caplin, Mark Dean, and John Leahy. Rational inattention, optimal consideration sets, and stochastic choice. *The Review of Economic Studies*, 86(3):1061–1094, 2018.
- Andrew Caplin, Mark Dean, and John Leahy. Rationally inattentive behavior: Characterizing and generalizing Shannon entropy. *Unpublished manuscript*, February 2019.
- Yeon-Koo Che and Konrad Mierendorff. Optimal dynamic allocation of attention. *American Economic Review*, 109(8):2993–3029, 2019.
- Nikolai Nikolaevich Chentsov. *Statistical decision rules and optimal inference*. Number 53. American Mathematical Soc., 1982.
- John A Clithero. Improving out-of-sample predictions using response times and a model of the decision process. *Journal of Economic Behavior & Organization*, 148:344–375, 2018.
- Thomas M Cover and Joy A Thomas. *Elements of information theory*. John Wiley & Sons, 2012.
- Henrique De Oliveira, Tommaso Denti, Maximilian Mihm, and Kemal Ozbek. Rationally inattentive preferences and hidden information costs. *Theoretical Economics*, 12:621–624, 2017.
- Mark Dean and Nathaniel Neligh. Experimental tests of rational inattention. *Unpublished manuscript*, June 2019.
- Ernst Fehr and Antonio Rangel. Neuroeconomic foundations of economic choice — recent advances. *Journal of Economic Perspectives*, 25(4):3–30, 2011.
- Drew Fudenberg, Philipp Strack, and Tomasz Strzalecki. Speed, accuracy, and the optimal timing of choices. *American Economic Review*, 108(12):3651–84, 2018.
- Benjamin Hébert and Michael Woodford. Information costs and sequential information sampling. *NBER Working Paper no. 25316*, November 2018.

- Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer Science & Business Media, 2013.
- Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6): 2590–2615, 2011.
- Ian Krajbich, Bastiaan Oud, and Ernst Fehr. Benefits of neuroeconomics modeling: New policy interventions and predictors of preference. *American Economic Review*, 104(5):501–506, 2014.
- Harold Kushner and Paul G Dupuis. *Numerical methods for stochastic control problems in continuous time*, volume 24. Springer Science & Business Media, 2013.
- Filip Matějka, Alisdair McKay, et al. Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review*, 105(1):272–98, 2015.
- Stephen Morris and Philipp Strack. The Wald problem and the relation of sequential sampling and ex-ante information costs. *Unpublished manuscript*, February 2019.
- Giuseppe Moscarini and Lones Smith. The optimal level of experimentation. *Econometrica*, 69(6): 1629–1644, 2001.
- Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 61. Springer Science & Business Media, 2009.
- Luciano Pomatto, Philipp Strack, and Omer Tamuz. The cost of information. *arXiv preprint arXiv:1812.04211*, 2018.
- Roger Ratcliff. Theoretical interpretations of speed and accuracy of positive and negative responses. *Psychological Review*, 92:212–225, 1985.
- Roger Ratcliff and Jeffrey N Rouder. Modeling response times for two-choice decisions. *Psychological Science*, 9:347–356, 1998.
- R Tyrrell Rockafellar. *Convex analysis*. 1970.
- JF Schouten and JAM Bekker. Reaction time and accuracy. *Acta Psychologica*, 27:143–153, 1967.
- Michael Shadlen and Daphna Shohamy. Decision making and sequential sampling from memory. *Neuron*, 90(5):927–939, 2016.
- Christopher A Sims. Rational inattention and monetary economics. *Handbook of Monetary Economics*, 3:155–181, 2010.
- Jakub Steiner, Colin Stewart, and Filip Matějka. Rational inattention dynamics: Inertia and delay in decision-making. *Econometrica*, 85(2):521–553, 2017.
- Satohiro Tajima, Jan Drugowitsch, and Alexandre Pouget. Optimal policy for value-based decision-making. *Nature communications*, 7, 2016.

Satohiro Tajima, Jan Drugowitsch, Nishit Patel, and Alexandre Pouget. Optimal policy for multi-alternative decisions. *Nature Neuroscience*, 22:1503–1511, 2019.

Cédric Villani. *Topics in optimal transportation*. Number 58. American Mathematical Soc., 2003.

Michael Woodford. Inattentive valuation and reference-dependent choice. *Unpublished manuscript*, May 2012.

Michael Woodford. Stochastic choice: An optimizing neuroeconomic model. Technical Report 5, 2014.

Michael Woodford. Modeling imprecision in perception, valuation and choice. *NBER Working Paper no. 26258*, September 2019.

Weijie Zhong. Indirect information measure and dynamic learning. *arXiv preprint arXiv:1809.00697*, June 2017.

Weijie Zhong. Optimal dynamic information acquisition. *Unpublished manuscript*, January 2019.

A Cost Functions Satisfying Conditions 1-5

In this appendix section, we discuss various families of cost functions that satisfy our conditions.

We first show that any posterior-separable cost function (in the terminology of Caplin et al. [2019]) that is sufficiently convex satisfies conditions 1-5. This includes all uniformly posterior-separable cost functions, such as mutual information and the neighborhood-based cost functions we describe in Hébert and Woodford [2018].

Posterior-separable cost functions are defined as

$$C(p, q; S) = \sum_{s \in S} \pi_s(p, q) D(q_s(p, q) || q),$$

where $D(\cdot || \cdot)$ is a divergence.

Lemma 8. *In the posterior-separable family of cost functions, if the divergence $D(\cdot || \cdot)$ is twice differentiable and strongly convex in its first argument, and differentiable in its second argument, the cost function $C(p, q; S)$ satisfies Conditions 1-5.*

Proof. See the appendix, section B.23. □

These posterior-separable cost functions a popular choice, but by no means the only possibility. As an alternative, consider “state-separable” cost functions,

$$C(p, q; S) = \sum_{x \in X} q_x D(p_x || \pi(p, q); S),$$

where $D(p_x || \pi(p, q); S)$ is a family of divergences defined on the signal alphabets S . Mutual information is both “posterior-separable” and “state-separable,” but in there are many cost functions in one family but not the other.

We will assume that the family of divergences is constant with respect to the addition of signals that never occur. That is, if $S \subset S'$, and $e_s^T p' = e_s^T p$ for $s \in S$ and zero otherwise, where e_s is a vector with one corresponding to signal s and zero otherwise, then

$$D(p_x || \pi(p, q); S) = D(p'_x || \pi(p', q); S').$$

We also assume a Blackwell-type inequality for these divergences,

$$D(\Pi r || \Pi r'; S') \leq D(r || r'; S)$$

for all r, r' and all garbling matrices $\Pi : S \rightarrow S'$. Note that this implies that D is invariant in the sense of Chentsov [1982]. Under these assumptions, and some regularity assumptions, we prove that this family also satisfies our conditions.

Lemma 9. *In the state-separable family of cost functions, if the divergences $D(\cdot || \cdot; S)$ are convex in their first argument and twice differentiable, and satisfies, for some $m > 0$ and all $r, r' \in \mathcal{P}(S)$ with*

$r' \ll r$,

$$D(r'|r; S) \geq m(r' - r)^T \text{Diag}(r)^+(r' - r),$$

then the cost function $C(p, q; S)$ satisfies Conditions 1-5.

Proof. See the appendix, section B.24. □

Lastly, we note that if some of cost function $C(p, q; S)$ satisfies our conditions, so does a convex transformation of that cost function, or a linear combination of two cost functions. Consequently, the two “separable” families described above can be used to generate a huge variety of non-separable cost functions.

Lemma 10. *If $C(p, q; S)$ is a family of cost functions satisfying Conditions 1-5, then so is $C_h(p, q; S) = h(C(p, q; S))$, where $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a strongly convex, strictly-increasing function with $h(0) = 0$. Likewise, if $C_1(p, q; S)$ and $C_2(p, q; S)$ are families of cost functions satisfying Conditions 1-5, then so is $C_\alpha(p, q; S) = \alpha C_1(p, q; S) + (1 - \alpha)C_2(p, q; S)$, for any $\alpha \in (0, 1)$.*

Proof. The proofs are almost immediate, given the the assumptions. □