REAL EFFECTS OF MONETARY SHOCKS IN AN ECONOMY WITH SEQUENTIAL PURCHASES

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1. Introduction

This paper is a theoretical study of the effects of monetary disturbances in an economy where firms set goods prices and consumers search over firms for the lowest priced goods. Each period, in this economy, cash transfers from the monetary authority determine consumers' ability to spend. The probability distribution of this transfer is common knowledge, but the realization is not known to sellers until trading is concluded. Exchange takes place sequentially within each period: sellers offer some goods for sale at prices they choose; consumers search over these priced goods, buying or not as they choose; sellers observe the sales that occur--their own, and others'--and on the basis of this information price additional goods, and so on. Trading ends when no consumer wishes to make additional purchases at the prices sellers post. After trading is concluded in a period, sellers' proceeds are recycled back to consumers, a new cash transfer is probabilistically determined, and the process continues, ad infinitum.

The equilibrium of this model exhibits a non-neutrality of money that is similar to many earlier models, one that we--in common with many earlier writers--believe may be an important feature of reality. Unanticipated changes in nominal spending flows induce less-than-proportional responses in nominal transaction prices, and changes in the same direction in real output. Since we take the economy's endowment--or its productive capacity--as fixed, periods of lower-than-average nominal spending are periods of excess capacity, of a kind that could not occur in Walrasian trading. Yet nothing prevents any seller from cutting his prices at any stage during the course of trading. Though it is possible to interpret the equilibrium of the model as one in which producers set prices in advance, neither nominal price commitments nor costs to changing nominal prices are assumed in the definition of an equilibrium.

The information structure of the game we study is, of course, crucial to the results we obtain, so the model has a surface resemblance to the model of Lucas (1972). The two models obviously have a similar motivation, and in both the inability of sellers to know the volume of spending that will occur in a period is central to the real effects of monetary transfers. Beyond this similarity, the two models have little in common. In Lucas (1972), sellers obtain information from prices that are set by a Walrasian mechanism; in this paper, prices are choice variables for sellers. In Lucas (1972), sellers in any one market are denied access to relevant information about the state of demand in other markets; in this paper, there is but one market, and every seller at every point in time has full information about everything that has occurred up to that time.

A much closer counterpart to the present paper is provided by Eden (1994). As is ours, Eden's model is a dynamic adaptation of a market game introduced by Prescott (1975) and Butters (1977), in which buyers search over goods that have been priced in advance by sellers. Eden obtains a Prescott-Butters-like equilibrium, as we do, in a game that does not require this element of advance commitment[1], and then embeds this one-period game in an intertemporal model of monetary exchange, as we do also. In the game studied by Eden, trading unfolds in a sequence of Walrasian markets, and non-price rationing plays no role. In ours, producer price-setting and rationing are both central features. It appears that the equilibrium obtained for a specific trading game by Prescott and Butters can be interpreted as describing the outcomes of a remarkable variety of trading games, in models that make very different assumptions on information, commitment, and the process of price setting. This makes it a useful abstract setting for trying to gain a deeper understanding of the elements that give rise to price rigidity and monetary non-neutrality.

Sections 2 and 3 of this paper define and analyze temporary equilibrium in a one-shot game of consumer search and producer pricing. This allows us to present the non-Walrasian market game while abstracting from dynamic complications. Section 4 then embeds the temporary equilibrium in a complete intertemporal monetary equilibrium, and proves the existence of a stationary equilibrium in the case of monetary shocks that are independent across periods. Section 5 discusses further the interpretation of our results, and concludes the paper.

2. A Pricing Game with Sequential Purchases

In this section we describe in some detail the kind of market game with sequential purchases that we have in mind. Since our theory centers on the progressive revelation of information through the process of trading, it is important to be explicit about the exact sequence of events. We describe a simple one-shot game, an adaptation of the games introduced by Prescott (1975) and Butters (1977).

There are two types of players, producers and consumers. We assume a continuum of length one for both player types. Each consumer i begins with θ dollars, where θ is a random variable with the probability measure Φ on an interval $\Theta = [\underline{\theta},\bar{\theta}] \subset \mathbb{R}_{++}$. He seeks to maximize $U(c) + \alpha s$, where c is total units of the good purchased and s is unspent cash remaining at the end of the game. We assume that U is continuously differentiable, strictly increasing, and strictly concave. The positive parameter α, the utility assigned to unspent cash, will be motivated in the multiperiod model developed in Section 4.

Each producer j begins with y units of a single, non-storable good. We may think of this as y units of inventory that can be sold, or as a capacity to produce up to y units. Producers know the distribution Φ of consumer cash holdings, but they do not observe the realization θ. Each producer's objective is to maximize the expected revenues from all units sold; unsold inventories (or unused capacity) have no value to producers.

We describe a game with a continuum of stages, taking place within a single period of real time, in which buyers acquire goods from sellers in exchange for cash at each stage. At each stage in this game, sellers offer goods for sale at prices they select. Consumers survey these priced goods, and buy from those that are priced lowest. Consumers competing for given units are rationed symmetrically. As goods are sold, producers are free to revise their offers for the current and subsequent stages without restriction. When no goods are offered for sale, or when no consumer wishes to buy, the game ends. All seller inventories remain unsold, and all buyer cash holdings remain unspent. We index the stages of this game continuously, by the cumulative number of units sold, z.

Let $c(\theta)$, $c:\Theta \rightarrow [0,y]$, denote the stage at which the game ends (aggregate quantity sold), which will in general depend on the shock θ. Let $p(z)$, $p:[0,y] \rightarrow \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$, be a function that describes the lowest price at which further goods are offered for sale at each stage z (that is, after an aggregate quantity z has already been sold). Here $p(z) = \infty$

means that no further goods are offered for sale at stage z. We require that $p(y) = \infty$, but our notation allows for the possibility that producers do not offer to sell even when unsold inventories exist. Since producers, who set prices, receive no information about θ during the play beyond that contained in the fact that the game has not yet ended (that $c(\theta) \geq z$) this function p will not depend on θ. All consumers and producers behave atomistically, which is to say they take these functions z and p as given. We will define a temporary equilibrium as a pair (c,p) that is consistent with optimal behavior of both buyers and sellers.

We require that $p(z)$ be right-continuous, to capture the idea that some goods must be placed on sale at a price in order for it to count as a price at which goods are offered for sale.[2] Because consumers will buy only the lowest priced goods at each stage (if they buy at all), $p(z)$ also indicates the price at which sales occur at stage z. Thus given a price function $p(z)$, we can define

$$R(z) = \int_0^z p(u)\,du \tag{2.1}$$

as the total spending by buyers (revenue to sellers) when z goods have been sold. (The integral is necessarily well-defined for all $0 \leq z \leq y$, though it may be infinite for large values of z.) Since p is taken as given by all players, so is R.

Note that $R(z)$ is non-decreasing, with $R(0) = 0$, and continuous for all $z < \bar{z} = \min\{z' \mid p(z') = \infty\}$. If no goods are ever sold at a price of zero--as will we show must be the case in equilibrium--then R is strictly increasing. In this case, we can invert R to express real output z as an increasing, continuous function of total spending θ. Hence if variation in θ causes variation in money spending, it <u>must</u> cause variation in real output; it cannot simply change transactions prices while leaving the quantity traded unaffected.

Buyers begin in identical situations, and all seek the lowest priced goods at each stage. We assume that these goods are rationed equally, so all buyers will acquire goods and deplete their cash at identical rates. Hence every buyer at stage z has acquired z units of the good and has $\theta - R(z)$ units of cash remaining. A buyer will cease to place orders only if his marginal utility of further consumption has fallen below the shadow value of the purchase price if added to end-of-period unspent cash (i.e., if $U'(z) > \alpha p(z)$);[3] <u>or</u> he has no remaining cash with which to make further purchases (i.e., if $R(z) = \theta$ and $p(z) > 0$). The latter condition may equivalently be described as a stage z such that $R(z') > \theta$ for all $z' > z$. Thus the game ends at stage z if one of the above conditions holds at z (or if one holds for all z' in a right neighborhood of z), while neither holds for any $z' < z$. Hence

$$c(\theta) = \inf\{z \mid U'(z) < \alpha p(z), \text{ or } R(z) > \theta\}.$$

Given the price function p, let

$$\hat{c} = \inf\{z \mid p(z) > \alpha^{-1} U'(z)\}. \tag{2.2}$$

Note that $\hat{c} \le \bar{z} \le y$. It then follows from the discussion above that

$$c(\theta) = \min[\max R^{-1}(\theta), \hat{c}] \, , \tag{2.3}$$

where $\max R^{-1}(\theta)$ denotes the maximal element in the set $\{z \mid R(z) = \theta\}$. (When the latter set is empty, we may define $\max R^{-1}(\theta) = \bar{z}$. Note that the exact value does not matter, as $c(\theta) = \hat{c}$ in any such case.) This consequence of buyer optimization shows how real output varies with the realization of θ. Below we establish that R is strictly increasing, so that $R^{-1}(\theta)$ is a single-valued function, continuous and monotonically increasing for all $\theta \le R(\bar{z})$. It then follows from (2.3) that $c(\theta)$ is a continuous function, monotonically increasing for all $\theta \le \hat{\theta} = R(\hat{c})$, and constant thereafter.

It will also be useful to define the function $\hat{p}(\theta)$ as:

$$\hat{p}(\theta) = \begin{cases} 0 & \text{if } \theta \le \hat{\theta} \\ \alpha^{-1}U'(\hat{c}) & \text{if } \theta > \hat{\theta} \end{cases} \, . \tag{2.4}$$

Here $\hat{p}(\theta)$ denotes the highest price at which buyers would be willing to buy more goods in the last stage of the game, if any were offered at a price that low. If $\theta \le \hat{\theta}$, all cash is spent, and buyers are unable to buy more at any positive price. If $\theta > \hat{\theta}$, buyers continue to hold cash after purchasing $\hat{c}$, and would be willing to buy more at any price not in excess of $\alpha^{-1}U'(\hat{c})$. Note that (2.2)-(2.4) imply that $\hat{p}(\theta) \le p(c(\theta))$ for all θ; otherwise, sales would not end at $c(\theta)$.

This describes the optimal behavior of buyers, given the function $p(z)$. We turn next to the sellers' problem. Each individual producer takes as given the functions $p(z)$, $c(\theta)$, and $\hat{p}(\theta)$ determined by the ensemble of strategies of all of the other players. At each stage z, each producer must post a price $p \in \bar{\mathbb{R}}_+$ at which he is willing to sell some of his endowment. Producers are free to change their posted prices continuously, as sales by themselves or others occur. A seller's price at any stage may depend upon aggregate sales z to that point and upon the units z_j that he has already sold. However, it cannot depend independently upon θ, for the seller's information about θ is always the same when stage z is reached. Nor need we allow for any dependence upon the details of the sequence of transactions that have occurred in the stages prior to z, for these are always the same when stage z is reached. We allow for dependence upon z_j in order to reflect the possibility that a producer may choose to sell his entire inventory at a given stage z, but not to offer all of it for sale at a single price. (For example, he may first offer $y/2$ for sale at a price p_1, and then his remaining $y/2$ for sale at a price $p_2 > p_1$, all of which units are sold before aggregate sales exceed z. We represent this by letting his price be p_1 when $z_j < y/2$, and p_2 when $y/2 \leq z_j < y$, all for a single value of z.) We also use the dependence upon z_j to express the constraint that a producer must post $p = \infty$ if $z_j = y$.

Units priced at $p(z)$ or lower will be sold at stage z (to be precise, if any sales in excess of z occur); units priced above this level will not. Thus it does not matter what price a seller posts at stage z if it exceeds $p(z)$; we may as well say that he chooses not to sell at that stage. Hence we can describe a seller's strategy by two functions, as follows.

(OUT OF ORDER)

Let $F_j(z_j)$ denote the aggregate quantity that will have been sold before the sales by j exceed z_j units. (More precisely, $F_j(z_j)$ is the supremum of values $z \le y$ such that j has sold z_j or less by stage z.) F_j maps $[0,y]$ into itself, and must be right-continuous and non-decreasing, with $F_j(y) = y$.

Second, let $p_j(z_j)$ be the price at which j will accept orders when he has already sold z_j units. Note that j might vary his price over a z-interval, but not obtain any orders and hence not increase his own sales z_j. But since no transactions occur at these prices, we do not need to specify what j's price offers are in such an interval. The price $p_j(z_j)$ is the price at which j actually fills his next order. Dependence of j's prices upon z need not be expressed, in the cases of prices at which j actually sells, because the function F_j already indicates the stage z corresponding to any quantity sold by j. The function p_j maps $[0,y]$ into $\bar{R}_+$, with $p_j(z_j) = \infty$. We also require p_j to be right-continuous. This means that a producer cannot post a price if he is not willing to sell a positive fraction of his endowment at prices in a neighborhood of that price, although we do allow the producer to vary the price continuously as he receives orders.

A pair of functions F_j and p_j satisfying the above requirements represents an admissable strategy for j if in addition

$$p_j(z_j) \le p(F_j(z_j))$$

for all $0 \le z_j \le y$. That is to say, at any stage at which j is in fact making sales, his price must not exceed the lowest price offered by other sellers. Note that our notation allows a producer to choose never to sell more than $y_j < y$ units. This can be represented by functions (F_j, p_j) such that $F_j(z_j) = y$ and $p_j(z_j) = \infty$ for all $z_j > y_j$.

This definition of the strategy space for seller j allows strategies of many kinds. For example, j may choose to accept orders in exact proportion to the aggregate quantity sold (so that $z_j = z$ at all times), and always to sell at the same price at which other transactions occur at that stage. Such a strategy is represented by the functions $F_j(z_j) = z_j$, $p_j(z_j) = p(z_j)$ for all z_j; in a symmetric equilibrium, all sellers choose strategies of this kind.

As another example, a seller may simply attach a price tag to each unit of inventory before trading begins, and offer all of the units for sale, never changing the price tags as trading proceeds. We can model such advance pricing as the choice of a measure π^j on the Borel sets of $\mathbb{R}_+$ that must satisfy $\pi^j(\mathbb{R}_+) \le y$, where $\pi^j(A)$ is the number of units with prices in a set A. Such a strategy is represented by the functions (F_j, p_j) such that

$$p_j(z_j) = \inf\{p \mid \pi^j([0,p]) > z_j\} \ ,$$

and

$$F_j(z_j) = \inf\{z \mid p(z) \ge p_j(z_j)\}$$

for every $0 \le z_j < y$. This is the kind of strategy that sellers must choose in the models of Prescott and Butters.

Given a strategy (F_j, p_j), producer j's sales in the case of state θ are given by

$$\text{(2.5)} \qquad z_j(\theta) = \sup\{z_j \mid F_j(z_j) \le c(\theta), \text{ or}$$

$$F_j(z_j) = c(\theta) \text{ and } p_j(z_j) \le \hat{p}(\theta)\} .$$

This indicates that sales occur beyond the point z_j if either a stage $z > F_j(z_j)$ is reached, or if the stage $z = F_j(z_j)$ is reached and $p_j(z_j) \le \hat{p}(\theta)$. In the latter case, sales beyond z_j occur even though aggregate sales equal only $F_j(z_j)$, because the price $p_j(z_j)$ is low enough that buyers are willing to buy more. Note that j's sales depend upon the functions $c(\theta)$ and $\hat{p}(\theta)$, but these are also taken as given in j's decision problem. Sales $z_j(\theta)$ result in revenues of

$$\text{(2.6)} \qquad r_j(\theta) = \int_0^{z_j(\theta)} p_j(y)dy \quad .$$

Then producer j's decision problem is to choose functions F_j and p_j to maximize expected revenues

$$\text{(2.7)} \qquad \int r_j(\theta) d\Phi(\theta)$$

given knowledge of the distribution Φ.

We require that the aggregate supply price function $p(z)$ and the ensemble of individual strategies $\{F_j(z_j), p_j(z_j)\}$ be consistent, in the following sense. For each j, the function F_j can be inverted to obtain a function $f_j(z)$ indicating the amount that j has sold by the time that stage z has been reached:

$$f_j(z) = \sup\{z_j \mid F_j(z_j) \le z\} .$$

Note that f_j is a right-continuous, non-decreasing function, taking $[0,y]$ into itself. Then we say that p is <u>consistent</u> with (F_j,p_j) if the inverse functions $\{f_j\}$ vary measurably with j, so that the integral $\int f_j(z)dj$ is well-defined for all $0 \le z \le y$;[4] if

$$\int f_j(z)dj = z \quad \text{for all} \quad 0 \le z \le y; \tag{2.8}$$

and if

$$p_j(z_j) = p(F_j(z_j)) \quad \text{for all} \quad 0 \le z_j \le y. \tag{2.9}$$

We can then define an equilibrium for the game as follows:

<u>Definition</u> A <u>temporary equilibrium</u> is a number $\hat{c}$ and a collection of functions $(p(z),c(\theta),R(z),\hat{p}(\theta))$, such that for some ensemble of producer strategies (F_j,p_j)

(i) given p, R is defined by (2.1);

(ii) given p and R, $\hat{c}$ is defined by (2.2), $c(\theta)$ by (2.3), and $\hat{p}$ by (2.4);

(iii) the aggregate supply price function p is consistent with the strategies $\{F_j, p_j\}$; and

(iv) for each j, (F_j, p_j) solves the producer's problem, given p, c, and $\hat{p}$.

In the next section we will prove that a unique temporary equilibrium exists, and characterize it.

3. Existence of a Temporary Equilibrium

Optimal consumer behavior has been characterized in equations (2.2) and (2.3) in the last section. We begin this section by further characterizing optimal pricing behavior by a producer. Let us consider the expected revenue $\rho(q,z)$ from from the sale of a particular unit of producer j's inventory, if this unit is to be sold only after stage z is reached, and if the price that j intends to charge in this event is q. (This is a feasible plan if and only if $q \leq p(z)$.) The unit is sold if either a stage beyond z is reached $(c(\theta) > z)$, or if $c(\theta) = z$ and $\hat{p}(\theta) \geq q$. Expected revenue from this unit of inventory is thus

$$\rho(q,z) = q\ \mathrm{Prob}\{c(\theta) > z\} + q\ \mathrm{Prob}\{c(\theta) = z,\ \hat{p}(\theta) \geq q\}\ .$$

It follows from (2.3)-(2.4) that $\hat{p}(\theta) > 0$ only if $\theta > \hat{\theta}$, in which case $c(\theta) = \hat{c}$. Hence for all $z < \hat{c}$,

$$\rho(q,z) = q\ \mathrm{Prob}\{c(\theta) > z\}\ .$$

It also follows from (2.3) that $\text{Prob}(c(\theta) > \hat{c}) = 0$, and so from (2.4) that

$$\rho(q,\hat{c}) = \begin{cases} q\ \text{Prob}(\theta > \hat{\theta}) & \text{if} \quad q \le \alpha^{-1}U'(\hat{c}) \\ 0 & \text{if} \quad q > \alpha^{-1}U'(\hat{c}) \end{cases}$$

Finally, it similarly follows from (2.3) that

$$\rho(q,z) = 0$$

for all $\hat{c} < z \le y$ (if this interval is non-empty).

Now given a choice of z (the stage at which the unit is to be sold), let us consider how expected revenues vary with q. We observe that in each of the three cases above $\rho(q,z)$ attains a maximum on the interval $0 \le q \le p(z)$. The maximum value is given by

$$\Lambda(z) = \max_{0 \le q \le p(z)} \rho(q,z)$$

$$(3.1) \qquad = \begin{cases} p(z)\ \text{Prob}(c(\theta) > z) & \text{if} \quad 0 \le z < \hat{c} \\ \alpha^{-1}U'(\hat{c})\text{Prob}(\theta > \hat{\theta}) & \text{if} \quad z = \hat{c} \\ 0 & \text{if} \quad \hat{c} < z \le y \end{cases} .$$

The set of maximizing prices is given by

$$Q(z) = \arg\max_{0 \le q \le p(z)} \rho(q,z)$$

$$= \begin{cases} \{p(z)\} & \text{if } 0 \le z < \hat{c} \\ \{\alpha^{-1}U'(\hat{c})\} & \text{if } z = \hat{c} \\ [0,p(z)] & \text{if } \hat{c} < z \le y \end{cases} .$$

Note that $\Lambda(z)$ is right-continuous on $[0,\hat{c})$ and constant on $(\hat{c},y]$; hence $\Lambda(z)$ is a measurable function on $[0,y]$. Also note that it is possible to make a right-continuous selection from the correspondence Q; for example, one might choose the function

$$q(z) = \min\{p(z),\ \alpha^{-1}U'(\hat{c})\} . \tag{3.2}$$

We now use the functions Λ and q and the correspondence Q to describe the optimal strategy for producer j, given the functions $p(z)$, $c(\theta)$, and $\hat{p}(\theta)$. We first consider the optimal choice of the function p_j, given a choice of the function F_j. Recall that the only constraint on p_j is that it be right-continuous and satisfy $0 \le p_j(z_j) \le p(F_j(z_j))$ for each z_j. The function

$$p_j(z_j) = q(F_j(z_j))$$

satisfies these requirements, for F_j any right-continuous, non-decreasing function, where $q(z)$ is defined in (3.2). Thus it is possible to choose p_j so that

$$p_j(z_j) \in Q(F_j(z_j)) \tag{3.3}$$

for each $z_j < y$, and so it is optimal to do so.

Next we consider the optimal choice of the function F_j, given that p_j is chosen to satisfy (3.3). It follows from (3.3) that expected revenues (2.7) are given by[5]

$$\int_0^y \Lambda(F_j(z_j))dz_j \ .$$

Producer j chooses F_j to maximize this expression, subject to the constraints that F_j be right-continuous and non-decreasing, and that $F_j(y) = y$. The latter constraints obviously do not prevent the producer from choosing a function such that

$$F_j(z_j) \in \arg\max_{0 \le z \le y} \Lambda(z)$$

for each $0 \le z_j < y$, and so it is optimal to do so.

Note that (3.1) implies that if $p(0) = \infty$, then $\Lambda(0) = \infty$, while $\Lambda(z) < \infty$ for all $z > 0$. Hence in this case, optimization requires that $F_j(z_j) = 0$ for all $0 \le z_j < y$. On the other hand, if $p(0) < \infty$, then $\Lambda(z)$ is bounded above. (Because $p(z)$ is right-continuous, there must exist an $\epsilon > 0$ and $\bar{p} < \infty$ such that $p(z) \le \bar{p}$ for all $0 \le z < \epsilon$. Then $\Lambda(z) \le p(z) \le \bar{p}$ on this interval. On the other hand, $\Lambda(z)$ is bounded above on $[\epsilon, y]$ as well, as noted above.) Then j's choice of F_j is optimal if and only if there exists a $\lambda > 0$ such that

$$\Lambda(F_j(z_j)) = \lambda \tag{3.4a}$$

for each $0 \le z_j < y$, while

(3.4b) $$\Lambda(z) \le \lambda$$

for all $0 \le z \le y$.

We next consider the conditions under which the aggregate supply schedule $p(z)$ is consistent with optimization by individual producers, each of whom takes the schedule $p(z)$ as given in choosing his own strategy (F_j, p_j). The following preliminary results establish some useful necessary conditions.

Lemma 1. Positive expected revenue is possible, so $\sup_z \Lambda(z) > 0$.

Proof: Let $\bar{z} = \inf\{z \mid p(z) > 0\}$. (Note that $\bar{z}$ must exist, as $p(y) = \infty$.) Then $p(z) = 0$ for all $z < \bar{z}$ (if any such exist), so that $\hat{c} \ge \bar{z}$, and $R(\bar{z}) = 0$.

Now if $\hat{c} = \bar{z}$, one must have $p(\bar{z}) > 0$. (This follows from (2.2) and the right-continuity of $p(z)$.) Then $\hat{\theta} = R(\bar{z}) = 0$, so that $\mathrm{Prob}\{\theta > \hat{\theta}\} = 1$. (Recall that we have assumed that $\theta \ge \underline{\theta}$ with probability one, for some $\underline{\theta} > 0$.) It follows that $\Lambda(\bar{z}) = \alpha^{-1}U'(\bar{z}) > 0$.

Alternatively, if $\hat{c} > \bar{z}$, then there must exist an interval $(\bar{z}, \hat{z})$, with $\bar{z} < \hat{z} < \hat{c}$, on which $p(z) > 0$. Furthermore, $\mathrm{Prob}\{c(\theta) > \bar{z}\} = \mathrm{Prob}\{\theta > 0\} = 1$. As $\mathrm{Prob}\{c(\theta) > z\}$ is right-continuous in z, one must also have $\mathrm{Prob}\{c(\theta) > z\} > 0$ for all z in a right neighborhood of $\bar{z}$. Thus there exists $\bar{z} < z < \hat{c}$ at which $\Lambda(z) = p(z)\mathrm{Prob}\{c(\theta) > z\} > 0$. □

Lemma 2. In any equilibrium, $\hat{c} = y$.

Proof: Because $p(y) = \infty$, (2.2) implies that $\hat{c} \leq y$. Hence we need only show that $\hat{c} \geq y$.

Suppose $\hat{c} < y$. Then (3.1) implies that $\Lambda(z) = 0$ for all $\hat{c} < z \leq y$. Because $\sup_z \Lambda(z) > 0$, optimization by producer j then requires that $F_j(z_j) \leq \hat{c}$ for all $0 \leq z_j < y$, which in turn implies that $f_j(z) = y$ for all $\hat{c} \leq z \leq y$. As this is true for all j,

$$\int f_j(z)dj = y$$

for all $\hat{c} \leq z \leq y$, which violates (2.8). Thus $\hat{c} = y$. □

Let us note some consequences of Lemma 2. First of all, it follows from (2.2) that $p(z) < \infty$ for all $z < y$. Second, it follows from (2.3)-(2.4) that for any state θ, either $c(\theta) = y$ (so that all goods are sold, capacity is exhausted) or $\hat{p}(\theta) = 0$ (so that buyers are unwilling to buy additional goods at any positive price). This observation is of some importance. It might be asked why our trading game does not allow producers to offer their remaining goods for sale after the state θ has been realized. But the equilibrium that we describe here would also be an equilibrium of a game with an additional stage of that kind. For once the stage $c(\theta)$ is reached, it is no longer possible for further exchange of money for goods to occur--either because sellers have no remaining goods, or because buyers have no remaining cash, or both.

Lemma 3. There exists a constant $\lambda > 0$ such that

$$(3.5) \qquad p(z) = \lambda[\mathrm{Prob}(c(\theta) > z)]^{-1}$$

for all $0 \le z < y$, while $p(y) = \infty$. Furthermore, λ satisfies

$$(3.6) \qquad \alpha^{-1}U'(y)\mathrm{Prob}(\theta > \hat{\theta}) \le \lambda .$$

<u>Proof</u>: It follows from Lemma 2 that $p(0) < \infty$, so that optimization by producer j requires (3.4a) and (3.4b). Now suppose that there exists $z_1 < y$ at which $\Lambda(z_1) < \lambda$. By the right-continuity of $\Lambda(z)$, there must also exist $z_1 < z_2 \le y$ such that $\Lambda(z) < \lambda$ for all $z_1 \le z < z_2$. But then (3.4a) requires that $F_j(z_j) \notin [z_1, z_2)$ for any $0 \le z_j < y$, which implies that $f_j(z)$ takes the same value for all $z_1 \le z < z_2$. As this must be true for every producer j,

$$\int f_j(z)dj$$

takes the same value for all $z_1 \le z < z_2$, which contradicts (2.8).

Thus one must have $\Lambda(z) = \lambda$ for every $0 \le z < y$. But then (3.1) and Lemma 2 imply (3.5) for every $0 \le z < y$. Furthermore, (3.4b) requires that $\Lambda(y) \le \lambda$. But then (3.1) and Lemma 2 imply (3.6). □

The result (3.5) has a number of interesting implications. It follows that $\mathrm{Prob}(c(\theta) > z) > 0$ for every $0 \le z < y$, and that $p(z) > 0$ for every $z \ge 0$. The latter result implies, as noted earlier, that $R(z)$ is monotonically increasing and hence invertible, so (2.3) becomes simply

$$(3.7) \qquad c(\theta) = \min[R^{-1}(\theta), \hat{c}] .$$

It also follows that $p(z)$ must be non-decreasing in z; as further purchases occur, prices may rise, but they never decline. Thus implies that $p(z) < \alpha^{-1}U'(z)$ for all $0 \le z < y$, so that buyers strictly prefer to buy at all stages prior to y, if they still have cash.

The result that $p(z)$ must be non-decreasing in z also implies that the equilibria of this model are also equilibria of a Prescott-Butters type search model, in which producers are restricted to strategies in which price tags are attached in advance to all units of inventory. For any aggregate supply price function $p(z)$ that is consistent with equilibrium is consistent with an equilibrium in which every producer j chooses a strategy of the form

$$F_j(z_j) = z_j ,$$

$$p_j(z_j) = p(z_j)$$

for all z_j. This strategy is a Prescott-Butters strategy,[6] corresponding to a price measure

$$\pi^j([0,p]) = \sup\{0 \le z \le y \mid p(z) \le p\} .$$

(Here we define the supremum as zero if the set is empty.) Because $p(z)$ is right-continuous and non-decreasing, $\pi^j([0,p])$ is well-defined for all $p \ge 0$, and is itself right-continuous and non-decreasing (as a cumulative distribution function must be).

With this characterization of producer optimization, we can now prove the existence of a temporary equilibrium (TE). For this purpose, it is convenient to use the shock distribution Φ to define a function $G:[0,\bar{\theta}] \rightarrow \mathbf{R}$ as follows:

$$G(\tilde{\theta}) = \int_0^{\tilde{\theta}} \mathrm{Prob}(\theta \geq \theta')d\theta' .$$

G is continuous, increasing, and concave, with $G(0) = 0$ and $G(\bar{\theta}) = E(\theta)$. Because G is concave, we can define its subdifferential $\partial G(\tilde{\theta})$:

$$\partial G(\tilde{\theta}) = \{g \in \mathbf{R} \mid G(\theta)-G(\tilde{\theta}) \leq g(\theta-\tilde{\theta}) \text{ for all } 0 \leq \theta \leq \bar{\theta}\}$$

$$= [\mathrm{Prob}(\theta > \tilde{\theta})\},\mathrm{Prob}(\theta \geq \tilde{\theta})\}] .$$

Note that for all $0 \leq \tilde{\theta} \leq \bar{\theta}$, $\partial G(\tilde{\theta})$ is a non-empty, closed, convex set, and the correspondence ∂G is non-increasing and upper-hemi-continuous. Moreover, $\partial G(0) = [1,\infty)$ and $\inf \partial G(\bar{\theta}) = 0$.

<u>Proposition 3.1</u> Let Φ, $U(\cdot)$, y, and $\alpha > 0$ be given. Then there exists a unique TE corresponding to each $0 < \hat{\theta} \leq \bar{\theta}$ such that

(3.8) $$\frac{\alpha}{yU'(y)} G(\hat{\theta}) \in \partial G(\hat{\theta})$$

and all TE are of this form.

Proof. We first show that for any TE, $\hat{\theta}$ satisfies (3.8). It follows from Lemma 2 that $p(z) \leq \alpha^{-1}U'(z)$ for all $z < y$. Because $p(z)$ is non-decreasing and $U'(z)$ is decreasing, the inequality is in fact strict. Using (3.5), this implies that

$$\frac{\lambda\alpha}{U'(z)} < \mathrm{Prob}(c(\theta) > z)$$

for all $z < y$, so that

$$\begin{aligned}\frac{\lambda\alpha}{U'(y)} = \sup_{z<y} \frac{\lambda\alpha}{U'(z)} &\leq \inf_{z<y} \mathrm{Prob}(c(\theta) > z) \\ &= \mathrm{Prob}(c(\theta) \geq y) \\ &= \mathrm{Prob}(\theta \geq \bar{\theta}) .\end{aligned}$$

Combining this with (3.6) yields

$$\frac{\lambda\alpha}{U'(y)} \in \partial G(\hat{\theta}) . \tag{3.9}$$

It also follows from (2.1), (3.7) and the fact that $p(c(\theta)) > 0$ for every $\theta \in \Theta$, that for any $\underline{\theta} \leq \theta \leq \hat{\theta}$,

$$c(\theta) = c(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{1}{p(c(\theta'))}\, d\theta' . \tag{3.10}$$

This in turn implies that

$$(3.11) \qquad y = c(\hat{\theta}) = \frac{1}{\lambda} G(\hat{\theta}) .$$

Consider first the case where $\underline{\theta} \le \hat{\theta}$. Then (3.5) implies that $p(z) = \lambda$ for all $0 \le z < c(\underline{\theta})$, so that $\underline{\theta} = R(c(\underline{\theta})) = \lambda c(\underline{\theta})$, and

$$c(\underline{\theta}) = \frac{1}{\lambda} \underline{\theta} = \frac{1}{\lambda} G(\underline{\theta}) .$$

Equation (3.5) also implies that $p(c(\theta')) = \lambda[\text{Prob}(\theta > \theta')]^{-1}$ for all $\underline{\theta} \le \theta' < \hat{\theta}$. Substitution of this into the second term on the right hand side of (3.10) yields $\frac{1}{\lambda}[G(\theta) - G(\underline{\theta})]$. Application to the case $\theta = \hat{\theta}$ then yields (3.11).

Consider next the case where $\hat{\theta} \le \underline{\theta}$. Then (3.5) implies that $p(z) = \lambda$ for all $0 \le z < y$, so that $\hat{\theta} = R(y) = \lambda y$, and

$$y = \frac{1}{\lambda} \hat{\theta} = \frac{1}{\lambda} G(\hat{\theta}) .$$

Thus (3.11) holds for this case as well. Substitution of (3.11) into (3.9) then yields (3.8).

Now let $\hat{\theta}$ be any solution to (3.8). We show that exactly one TE can be constructed for this value of $\hat{\theta}$. Let $\lambda^* = G(\hat{\theta})/y$. Since $\hat{\theta} > 0$, $\lambda^* > 0$. Then define $c(\theta)$ by

$$c(\theta) = \frac{1}{\lambda^*} G(\theta) \qquad \text{for all } \underline{\theta} \le \theta \le \hat{\theta} ,$$

$$c(\theta) = y \qquad \text{for all } \theta \ge \hat{\theta} .$$

Then c is a continuous, non-decreasing, non-negative function, monotonically increasing for all $\underline{\theta} < \theta < \hat{\theta}$. (If $\hat{\theta} \leq \underline{\theta}$, these conclusions hold vacuously.)

Define $\hat{p}(\theta)$ by (2.4). Then define $p(z)$ by (3.5), setting $\lambda = \lambda^*$ and defining $p(y) = \infty$. Note that $p(z)$ is a positive, right-continuous function. The continuous, increasing function $R(z)$ is then given by (2.1). We can choose an ensemble of strategies (F_j, p_j) for individual producers to be consistent with $p(z)$ in various ways. One simple choice is to set

$$F_j(z_j) = z_j ,$$

$$p_j(z_j) = p(z_j)$$

for all $0 \leq z_j \leq y$, for every j.

Clearly, the value λ^* and the functions $c(\theta)$, $\hat{p}(\theta)$, $p(z)$, and $R(z)$ are uniquely defined, and these are the only functions that can correspond to a TE. Moreover, it is easily seen that the functions just constructed satisfy all the requirements for a TE. It follows from the facts that $\hat{\theta}$ satisfies (3.8) and that $p(z)$ is defined by (3.5) that $\hat{c} = y$ in (2.2). Then the construction of $c(\theta)$ and $\hat{p}(\theta)$ above guarantees that (2.3) and (2.4) are satisfied for $\hat{c} = y$; thus $(c,\hat{p})$ represents optimal buyer behavior given (p,R). The aggregate supply function $p(z)$ is clearly consistent with the ensemble of strategies (F_j, p_j). Finally, (F_j, p_j) is optimal for each producer j, given $(p,c,\hat{p})$. For (3.5) implies that $\Lambda(z) = \lambda$ for all $0 \leq z < y$, from which it follows that any choice of F_j such that $0 \leq F_j(z_j) < y$ for all $0 \leq z_j < y$ is optimal. The choice of p_j

is optimal if and only if (3.3) is satisfied for each $0 \le z_j < y$, and this is so in the above construction. □

<u>Proposition 3.2</u> For any $\alpha > 0$, there exists a unique TE, corresponding to a unique $\hat{\theta}(\alpha) > 0$. Furthermore, $\hat{\theta}(\alpha)$ is a continuous, non-increasing function.

<u>Proof</u>. To prove the existence of a unique TE, we need only show the existence of a unique solution to (3.8) for each $\alpha > 0$. Since $G(\theta)$ is continuous and strictly increasing in θ, $\frac{\alpha}{yU'(y)}G(\theta)$ has these same properties as a function of θ. On the other hand, $\partial G(\theta)$ is non-increasing in θ, with a closed graph, and the range $[0,\infty)$. Hence there is a solution $0 \le \hat{\theta} \le \bar{\theta}$ to (3.7) for any $\alpha > 0$. Furthermore, since

$$\frac{\alpha}{yU'(y)}\, G(0) = 0 < \inf \partial G(0)$$

all solutions must satisfy $\hat{\theta} > 0$. If there are two solutions $\hat{\theta}_1 < \hat{\theta}_2 \le \bar{\theta}$, then one must have $G(\hat{\theta}_1) = G(\hat{\theta}_2)$, which would contradict the fact that G is monotonically increasing for all $\theta \le \bar{\theta}$.

Thus the function $\hat{\theta}:\mathbb{R}_{++} \to (0,\bar{\theta}]$ is well-defined, and we consider its behavior as α varies. Let $\alpha_2 > \alpha_1 > 0$. Then $\hat{\theta}(\alpha_1)$ is the unique θ for which (3.8) holds, and hence for all $\theta > \hat{\theta}(\alpha_1)$, $\frac{\alpha_1}{yU'(y)}G(\theta) > \sup \partial G(\theta)$. Since $\alpha_2 > \alpha_1$, it follows that $\frac{\alpha_2}{yU'(y)}G(\theta) > \sup \partial G(\theta)$ for all $\theta > \hat{\theta}(\alpha_1)$. Then (3.10) cannot be satisfied at α_2 for any $\theta > \hat{\theta}(\alpha_1)$, which proves that $\hat{\theta}(\alpha_2) \le \hat{\theta}(\alpha_1)$, or that $\hat{\theta}(\alpha)$ is non-decreasing.

We next show that $\hat{\theta}(\alpha)$ is continuous for all $\alpha > 0$. For some sequence (α_n) with $\alpha_n > 0$ for each n and $\alpha_n \to \alpha^* > 0$, suppose that $\theta_n = \hat{\theta}(\alpha_n)$ for each n, and that $\theta^* = \hat{\theta}(\alpha^*)$. We wish to show that $\theta_n \to \theta^*$. That is, for any θ' such that $\theta' < \theta^*$, we wish to show that $\theta' < \theta_n$ for all n large enough, and similarly, for any θ'' such that $\theta'' > \theta^*$, we wish to show that $\theta'' > \theta_n$ for all n large enough. Consider first the lower bound. Since $\alpha_n \to \alpha$ and $G(\theta^*) > G(\theta')$, the inequality

$$\frac{G(\theta^*)}{G(\theta')} \alpha > \alpha_n$$

must hold for n sufficiently large. We show that this inquality in turn implies that $\theta_n > \theta'$. For if one had $\theta_n \le \theta'$, one could show that

$$\begin{aligned} \inf \partial G(\theta_n) &\le \frac{\alpha_n}{yU'(y)} G(\theta_n) \le \frac{\alpha_n}{yU'(y)} G(\theta') \\ &< \frac{\alpha^*}{yU'(y)} G(\theta') \le \sup \partial G(\theta^*) \le \inf \partial G(\theta_n) , \end{aligned}$$

which is a contradiction. The proof for the upper bound θ'' is identical.
□

4. Stationary Equilibrium with Independent Monetary Shocks

We now embed the temporary equilibrium of the previous section in a complete intertemporal equilibrium. We consider an economy operating in an infinity of periods $t = 0,1,2,\ldots$ In each period, consumers exchange cash for goods provided by sellers in exactly the manner described in Section 2.

Cash acquired by sellers is returned to consumers at the end of the period, after the pricing game is completed, in the form of a dividend to shareholders. The monetary authority uses beginning-of-period and end-of-period transfers in such a way as to make consumer cash holdings θ_t an independent, identically distributed random variable. In this section we spell out the details of this economy, and show that its stationary equilibrium can be obtained by the same construction used to characterize the temporary equilibrium in Section 3. The idea will be to show that the value that consumers assign to unspent cash, the given parameter α in the preceding sections, can be reinterpreted as a marginal value derived from consumers' intertemporal maximum problem.

The economy we consider is made up of a continuum of infinite-lived households, each of which seeks to maximize the expected value of

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{4.1}$$

where U is the same single-period utility function as above, and β is a discount factor in (0,1). Note that there is no direct utility from cash balances of the kind assumed in Sections 2 and 3. The monetary authority is assumed to choose a monetary injection θ_t in each period t, the value of which is again not known to producers until after all period t sales have occurred. The state variable (θ_t) is assumed to be identically and independently distributed across periods, being drawn each period from the same distribution as above. Hence the information of producers at the beginning of period t is simply that θ_t will be drawn from the distribution Φ on $\Theta = [\underline{\theta},\bar{\theta}] \subset \mathbf{R}_{++}$, as in the temporary equilibrium of the previous section. In this section, we also assume that $\underline{\theta} < 1$.

In order to calculate the endogenously determined value of end-of-period cash balances, we must consider how a given consumer's budget constraint in the following period will be affected if he deviates from equilibrium behavior. Let us return again to the consumer search process described in Section 2. There we assumed that all consumers entered each stage of the game in identical positions, and that goods were rationed among them symmetrically. Here we consider the possibility that a single (non-atomic) consumer i will, as a result of his actions in the past, begin a period with money holdings $\mu\theta$, for some $\mu > 0$, while all other consumers have θ. As observed in Section 2, each consumer will purchase as many units as he can at each stage prior to the last stage in which he buys any positive quantity. We assume that at each stage prior to the final stage $z = c(\theta)$, consumers are rationed in proportion to their unspent cash holdings, so that consumer i is rationed μ times the goods allocated to anyone else. Thus when z goods have been sold, if $z < c(\theta)$ and i has not stopped buying, consumer i will have obtained μz units at a cost of $\mu R(z) = \mu\int_0^z p(u)du$ dollars. At the final stage, we assume that consumer i can purchase whatever he wants at the final price $p(c(\theta))$. The expenditure he requires to purchase c units of consumption is thus

$$\mu\int_0^{c/\mu} \min[p(z),p(c(\theta))]dz \ .$$

Now let m_t^i denote consumer i's money balances at the beginning of period t, before the period t monetary injection occurs. Each consumer receives a lump-sum transfer of $\theta_t - 1$ at the beginning of period t, and a lump-sum transfer of τ_t at the end of the period. At the end of the period, each consumer also receives as a dividend his proportional share of the total sales revenues Π_t of the firms. Then if consumer i buys c_t^i units of goods, he will begin the next period with money balances of

$$m_{t+1}^i = \mu_t^i(\theta_t - \int_0^{c_t^i/\mu_t^i} \min[p(z), p(c(\theta_t))]dz) + \Pi_t + \tau_t , \tag{4.2a}$$

where

$$\mu_t^i = \theta_t^{-1}[m_t^i + \theta_t - 1] . \tag{4.2b}$$

The end-of-period lump-sum transfer is assumed to be

$$\tau_t = 1 - \theta_t . \tag{4.2c}$$

Finally, since in a symmetric equilibrium the revenues Π_t equal the expenditure of each consumer, if $m_t^i = 1$ then $m_{t+1}^i = 1$ as well. We thus consider an equilibrium in which $m_t^i = 1$ forever, in which θ_t again represents the money supply in period t, and the expression in (4.2b) indicates the ratio of i's post-injection cash balances to those of the typical consumer.

The problem of consumer i is then to choose a plan specifying consumption purchases c_t^i in each period as a function of the history of realizations $(\theta_0, \ldots, \theta_t)$ of the monetary injections, so as to maximize the expected value of (3.1), subject to the constraints that period t expenditure not exceed $\mu_t\theta_t$, and that $\mu^i_{t+1} \geq 0$, in all periods and under all possible histories of monetary injections, given the laws of motion (4.2). Here the determination of Π_t as a function of the history of monetary injections is also taken as given by the consumer, and similarly the functions p and c, which may also depend upon the history of monetary injections. Initial money balances $m^i_0 = 1$ for all i are given as an initial condition.

We now specialize to the case of a stationary equilibrium, in which the functions (p,c) are the same for all t. We also assume that each period $\Pi_t = \Pi(\theta_t)$, where the function $\Pi(\theta)$ is also the same for all t. The consumer's problem then takes a stationary recursive form. Let $v(m^i_0)$ denote the maximum attainable value for the expected value of (4.1) given the constraints listed above, for any initial money balances $m^i_0 \geq 1-\underline{\theta}$. This value function v must satisfy the Bellman equation

$$(4.3a) \qquad v(m) = \int [\max_{c,\tilde{m}} \{U(c) + \beta v(\tilde{m})\}]\Phi(d\theta)$$

where for each $m \geq 1-\underline{\theta}$, $\underline{\theta} \leq \theta \leq \bar{\theta}$, $(c,\tilde{m})$ must be chosen such that

$$(4.3b) \qquad \max[1-\underline{\theta},\Pi(\theta)+\tau(\theta)] \leq \tilde{m} \leq m + \theta - 1 + \Pi(\theta) + \tau(\theta) ,$$

$$(4.3c) \qquad \tilde{m} = \mu(m,\theta)\{\theta - \int_0^{c/\mu(m,\theta)} \min[p(z),p(c(\theta))]dz\} + \Pi(\theta) + \tau(\theta)$$

$$\text{if } \mu(m,\theta) > 0 ,$$

(4.3d) $$c = 0 \quad \text{if} \quad \mu(m,\theta) = 0 .$$

Here $\mu(m,\theta)$ is the function defined in (4.2b) and $\tau(\theta)$ is the function defined in (4.2c). Equation (4.3c) restates the law of motion (4.2a). The lower bound $1-\underline{\theta}$ in (4.3b) follows from the requirement that $\mu^i_{t+1} \geq 0$ regardless of the realization of θ_{t+1}. The lower bound $\Pi(\theta) + \tau(\theta)$ follows from the requirement that the household's expenditure not exceed $\mu_t\theta_t$, as does (4.3d). Finally, the upper bound in (4.3b) follows from the requirement that $c^i_t \geq 0$.

This recursive formulation allows us to describe consumer behavior each period by a choice of total purchases $c^i_t = c(m^i_t,\theta_t)$, where $(c(m,\theta),\tilde{m}(m,\theta))$ is the solution to the maximization problem inside the brackets in (4.3a). This will in turn allow us to describe a stationary intertemporal equilibrium as a succession of temporary equilibria of the kind analyzed in the previous section. But since a stationary equilibrium involves the value function $v(m)$, before we can define a stationary equilibrium we need to ensure the existence of a solution to (4.3) (Proposition 4.1) and to establish its differentiability at $m = 1$ (Proposition 4.2). These results develop some other properties of v and the associated policy functions $(c,\tilde{m})$ as well.

<u>Proposition 4.1</u>. In the maximization problem (4.3), suppose that

(i) the function $c(\theta)$ is continuous on Θ;

(ii) the function $p(z)$ is non-decreasing and right-continuous, with $p(z) > 0$ for all $0 \leq z \leq y$;

(iii) the function $\Pi(\theta)$ is continuous, with $\Pi(\theta) \geq 0$ for all $\theta \in \Theta$; and

(iv) the function $U(c)$ is bounded, continuous, strictly increasing, and concave, for all $c \geq 0$.

Then there exists a value function $v(m)$, defined for all $m \geq 1-\underline{\theta}$, that satisfies the Bellman equation (4.3a). Furthermore, $v(m)$ is bounded, continuous, strictly increasing, and concave.

Proof: See the appendix.

By a standard argument (see, e.g., Stokey, Lucas, and Prescott, 1989, Theorem 9.2), the existence of a function $v(m)$ that satisfies the Bellman equation implies the existence of a solution to the original infinite-horizon consumer optimization problem, and optimal behavior consists of choosing each period

$$c_t^i = c(m_t^i, \theta_t) ,$$

$$m_{t+1}^i = \tilde{m}(m_t^i, \theta_t) ,$$

where again $(c, \tilde{m})$ are the functions that solve the optimization problem inside the brackets in (4.3a). We accordingly turn our attention to that problem.

In a stationary equilibrium, we want $c(\theta) = c(1,\theta)$ to hold, and to specify $\Pi(\theta)$ as

$$\Pi(\theta) = R(c(\theta)) = \int_0^{c(\theta)} p(z)dz . \tag{4.4}$$

(Note that we have not yet proved that this is possible, since we have only shown the existence of a function $c(1,\theta)$ given some specification of $c(\theta)$ and $\Pi(\theta)$; we do not know that we can find functions $c(\theta)$ and $\Pi(\theta)$ such that the $c(1,\theta)$ that attains (4.3a) will equal $c(\theta)$ and also generate that same function $\Pi(\theta)$ in (4.4).) Then given the initial condition $m_0^i = 1$ for all i, each household's optimal behavior will be to choose

$$c_t^i = c(\theta_t) \;,$$

$$m_{t+1}^i = 1 \;,$$

in all periods. (This follows from (4.3c).)

Hence it suffices to consider the single period maximization problem in (4.3a) for the case $m = 1$. Suppose that, as assumed in the previous section, $U(c)$ is continuously differentiable for all $c > 0$, strictly concave, and that $\lim_{c \to 0} U'(c) = +\infty$. Suppose also that $v(m)$ is differentiable at $m = 1$. Then necessary and sufficient conditions for the function $c(\theta)$ to solve the problem in (4.3a) (given that $R(\theta)$ satisfies (4.4)) are that, for each $\theta \in \Theta$, $c(\theta)$ be the infimum of the set of z such that either

$$U'(z) < \beta v'(1)p(z) \tag{4.5a}$$

or

$$\int_0^{c(\theta)} p(z)dz > \theta \;. \tag{4.5b}$$

If the parameter α in Sections 2 and 3 is given the value

$$(4.6) \qquad \alpha = \beta v'(1) ,$$

this case is seen to correspond exactly to the characterization of $c(\theta)$ in Section 2, and in particular to imply that $c(\theta)$ is given by (3.7), where $\hat{c}$ is again given by (2.2). Thus with the identification (4.6) of α, consumer behavior each period in a stationary equilibrium is exactly optimal consumer behavior in the TE characterized in the previous section.

Sufficient conditions for the differentiability of the value function, and hence for the characterization (4.5) of consumer behavior, are given by the following result.

Proposition 4.2. Assume hypotheses (i)-(iv) of Proposition 4.1, and suppose that $c(1,\theta) = c(\theta)$ solves the maximum problem (4.3) at $m = 1$. Suppose in addition that

(v) $c(\theta)$ is uniformly bounded away from zero on Θ;

(vi) the function $U(c)$ is continuously differentiable for all $c > 0$, it is strictly concave (i.e., $U'(c)$ is monotonically decreasing), and $\lim_{c \to 0} U'(c) = +\infty$; and

(vii) $\Pi(\theta)$ is given by (4.4).

Then the value function $v(m)$ that satisfies (4.3a) is differentiable at $m = 1$, and the derivative equals

$$(4.7) \qquad v'(1) = (1-\beta)^{-1}\int \theta^{-1} U'(c(\theta))c(\theta)\Phi(d\theta) \; .$$

Proof. See the appendix.

Hence the characterization of optimal consumer behavior in the previous section continues to apply to a stationary equilibrium, with the identification of the parameter $\alpha > 0$ given by (4.6) and (4.7). We next consider optimal producer behavior. Each producer j has a capacity constraint $y > 0$ in each period. The producer chooses an admissable strategy (F_j, p_j), as defined in Section 2, yielding sales as a function of θ as given by (2.5) and revenues as given by (2.6).

Producers distribute all earnings to the households (owners of the firms) at the end of the period, and producer j chooses (F_{jt}, p_{jt}) so as to maximize the value to the representative household of an increment in its earnings distribution, contingent upon θ_t, of the form $r_{jt}(\theta_t)$. The value to be maximized is an ex ante value, before the value of θ_t is known, and firm j takes as given the aggregate earnings distribution function $R(c(\theta_t))$ in calculating the value of an incremental distribution. Then, under the hypotheses of Proposition 4.2, (F_{jt}, p_{jt}) is chosen so as to maximize

$$\int r_{jt}(\theta)v'(\bar{m}(m_t, \theta))\Phi(d\theta) = v'(1)\int r_{jt}(\theta)\Phi(d\theta) \; .$$

That is, it is chosen so as to maximize expected revenues, given that θ_t will be drawn from the distribution Φ. Thus (F_j, p_j) is chosen each period in exactly the way assumed in the temporary equilibrium of the previous section, and the characterization of optimal producer behavior there applies here as well.

Under the hypotheses of Proposition 4.2, then, the characterizations of both producer and consumer behavior in the previous section continue to apply. As a result, a stationary equilibrium will involve a succession of temporary equilibria of the kind described earlier. But our previous characterization of temporary equilibrium implies that the hypotheses of Proposition 4.2 will indeed hold in such a case. Hence we may define a stationary equilibrium as follows:

Definition. A stationary equilibrium is a constant $\alpha > 0$, a number $\hat{c}$, and a collection of functions $(p(z), c(\theta), R(z), \hat{p}(\theta))$ such that

(i) given $\alpha > 0$, $(\hat{c}, p, c, R, \hat{p})$ constitute a temporary equilibrium in the sense defined in Section 2; and

(ii) the constant α satisfies

$$(4.8) \qquad \alpha = \beta(1-\beta)^{-1}\int \theta^{-1} U'(c(\theta))c(\theta)\Phi(d\theta) \ .$$

Here (4.8) follows from (4.6) and (4.7).

The following result shows that such an equilibrium exists for all possible distributions of the monetary shocks.

Proposition 4.3. Suppose that the function $U(c)$ is bounded, continuous, strictly increasing, and strictly concave for all $c \geq 0$, continuously differentiable for all $c > 0$, and satisfies $\lim_{c \to 0} U'(c) = +\infty$. Then there exists a stationary equilibrium.

Proof. By Proposition 3.2, there is a unique TE for every $\alpha > 0$, and $\hat{\theta}(\alpha)$, the unique solution to (3.8), is a continuous, non-increasing function of α, with $\hat{\theta}:\mathbf{R}_{++} \rightarrow (0,\bar{\theta}]$. It remains only to show that α can be chosen so that (4.8) is satisfied.

To this end, we next consider how the function $c(\theta;\alpha)$ in a TE with given α varies with α. For any $\alpha > 0$, the construction used in the proof of Proposition 3.1 implies that

$$c(\theta;\alpha) = y \min[\frac{G(\theta)}{G(\hat{\theta}(\alpha))}, 1] .$$

Now for pairs (θ,α) such that $\theta \in \Theta$, $\alpha > 0$, and $\hat{\theta}(\alpha) \geq \theta$, the continuity of $G(\theta)$, the continuity of $\hat{\theta}(\alpha)$, and the fact that $G(\hat{\theta}(\alpha)) \geq G(\theta) \geq G(\underline{\theta}) > 0$ imply that $G(\theta)/G(\hat{\theta}(\alpha))$ is a continuous function of (θ,α). Moreover, this function equals 1 at all points on the boundary where $\hat{\theta}(\alpha) = \theta$. Hence $c(\theta;\alpha)$ is a continuous function of (θ,α) on the domain $\Theta\times\mathbf{R}_{++}$. Furthermore, if one defines

$$c(\theta;0) = y\, G(\theta)/G(\underline{\theta})$$

then $c(\theta;\alpha)$ is a continuous function on the domain $\Theta\times\mathbf{R}_{+}$.

The function $c(\theta;\alpha)$ is obviously non-decreasing in both arguments on that domain. Furthermore, one observes that for any $\theta \in \Theta$, $\lim_{\alpha\rightarrow\infty} c(\theta;\alpha) = y$. Finally, the function is bounded and bounded away from zero on $\Theta\times\mathbf{R}_{+}$, insofar as

$$0 < y\, G(\underline{\theta})/G(\bar{\theta}) \leq c(\theta;\alpha) \leq y$$

for all $\underline{\theta} \le \theta \le \overline{\theta}$, $\alpha \ge 0$.

Now consider the right hand side of (4.8), as a function of α. Because $U(c)$ is continuously differentiable for all $c > 0$, and $c(\theta;\alpha)$ is continuous in both arguments, bounded, and bounded away from zero, the function

$$\frac{c(\theta;\alpha)U'(c(\theta;\alpha))}{\theta}$$

is a continuous function of (θ,α) on the domain $\Theta\times\mathbb{R}_+$ that is both bounded and bounded away from zero. Then by the Lebesgue dominated convergence theorem, the integral of this function over Θ is a continuous function of α on the domain $\alpha \ge 0$, and is both bounded and bounded away from zero. Hence the right hand side of (4.8) is a function of α with these properties.

It follows that both the left hand and right hand sides of (4.8) are continuous functions of α, with the left hand side necessarily larger for large enough α, and smaller for small enough $\alpha > 0$. Hence there must exist a solution for some $\alpha > 0$. Given this value, the $(\hat{c},p,c,R,\hat{p})$ that describe the temporary equilibrium for this value of α then constitute a stationary equilibrium. □

5. Discussion

The stationary equilibrium we have just constructed consists of a sequence of temporary equilibria of the kind characterized in section 3. In each period t there is another independent drawing of the shock θ_t that determines the period's money supply. The aggregate purchases that result are given by $c_t = c(\theta_t)$, and the marginal price level (the highest price at which all goods offered for sale are sold) is given by $p_t = p(\theta_t)$. The distribution of transaction prices in period t is given by the function $p(z)$ on the interval $[0,c(\theta_t)]$, and total nominal spending is given by $R(c(\theta_t)) = \min(\theta_t,\hat{\theta})$. The i.i.d. random variations in θ_t thus give rise to i.i.d. variations in spending, consumption, and prices.

Under the interpretation in which producers begin with a productive capacity rather than an endowment of goods and produce only to fill the orders that they accept, the fluctuations in c_t represent fluctuations in real output. As promised, then, we have exhibited a model in which surprise variations in the money supply affect not only nominal spending and prices, but real activity as well. Because $c(\theta)$ is a non-decreasing function, monotonically increasing over the range $\underline{\theta} \le \theta \le \hat{\theta}$, low realizations of the money supply are associated with low levels of output.[7] The model can thus rationalize the association between low rates of growth of the money supply and contractions of economic activity documented by Friedman and Schwartz (1963) and many others. The marginal price level is also non-decreasing in θ, so that higher realizations of the money supply are generally associated with higher prices being reached. The <u>average</u> transaction price, $\frac{1}{c}\int_0^c p(z)dz$, is non-decreasing in θ and increasing in c. Thus the model predicts an upward-sloping Phillips-curve relation linking output movements to corresponding movements of average transactions prices.

It is also worth noting that in this model uncertainty about the money supply reduces the <u>average</u> level of output as well as increasing its variability. With a fully predictable money supply, output equals capacity at all times. With stochastic money shocks, output is sometimes at capacity and sometimes below it. The model thus captures an association between inflation uncertainty and real activity that is often asserted to exist.

The model we have analyzed has a fixed endowment of goods (or capacity) but it is easy to imagine variations in which goods are produced by labor, some of which is contracted for prior to the realization of θ, and the rest of which is hired on a spot market during the course of trading. In such an elaboration of the model, a high demand shock would induce both high production per unit of labor under contract and a large employment of spot labor. Even though the production technology is not actually changing, monetary surprises will produce variation in the measured Solow residual, correlated with output and employment variations. This kind of effect is discussed in detail by Eden (1990), Rotemberg and Summers (1990), and Eden and Griliches (1993), in the context of non-monetary models of the Prescott-Butters type.

In this model, as in many earlier models, variations in the money supply affect real activity only because they are <u>unanticipated</u>. If, by way of contrast, the drawings of θ_t were made public before any trading occurred, consumers would still purchase goods until the marginal price of goods reached the value $\alpha^{-1}U'(c)$, but producers would be able to choose strategies contingent upon θ_t. Since there would be no uncertainty about the marginal price p_t, each producer would offer to sell y units at the price p_t, and none at any lower price. The equality $c(\theta) = y$ would hold for all θ, and the equilibrium price would be given by

$$p(\theta) = \min[\frac{\theta}{y}, \frac{U'(y)}{\alpha}] .$$

The resulting equilibrium would in fact be identical to the equilibrium of a cash-in-advance model with Walrasian spot markets (see, e.g., Sargent, 1987, chap. 5).[8]

In the present model, as in that of Lucas (1972), producers' imperfect information about the state of nominal aggregate demand is crucial to the non-neutrality of monetary shocks. But in Lucas (1972), the real effects of monetary shocks depend upon producers' ignorance about the shadow value to them of the cash they acquire in current trading. In a recession, sellers supply too little because they fail to realize how low prices generally will be in the following period, when they spend the cash obtained from current sales. In the present model, by contrast, producers correctly understand the shadow value of cash ($\beta v'(1)$ in our notation above), and knowledge of the current shock θ_t would not affect their evaluation of this value. The information that they lack, instead, is about how the price they charge will affect the quantity that they will be able to sell. In a recession state, producers offer to supply too few goods at low money prices, because they overestimate the chance of eventually finding buyers who will pay a high price; if they knew the current state of nominal aggregate demand they would know better.

Of course, if we were to drop our unrealistic assumption that the money supply is independently distributed across periods, and assume instead that the money shocks are serially correlated, then a seller's estimate of θ_t would affect his estimate of $\beta v'(1)$, and misperception of the shadow value of cash due to ignorance of the current money supply would indeed bear some of the blame for the inefficient use of resources associated with monetary instability. But a still more realistic model would recognize that each producers' sales are stochastic for many reasons independent of the current realization of money growth, and as a result that there is little reason for producers to revise their estimates of either the current or the future money supply on the basis of surprise variations in their own current sales. Consider, for example, a model with many submarkets each organized along the non-Walrasian lines explained in Section 2, but with independent variations in spending in the various submarkets superimposed upon the (possibly autocorrelated) variation in aggregate spending. We would expect each submarket's equilibrium to be similar to the kind described above, with producers choosing a pricing strategy similar to the one that would maximize expected revenues, even in the case of significant persistence in the fluctuations in aggregate spending.

There is an alternative interpretation of the equilibrium of Section 4, under which consumers--symmetrically with firms--learn the current realization θ_t only in the course of trading. Instead of assuming that θ_t is given to consumers as a single, beginning-of-period transfer, one can assume instead that consumer cash balances are $\underline{\theta}$ at the beginning of trading in all periods, and that additional transfers occur continuously during trading, stopping when a total money supply of θ_t per household has been reached. The drawing of θ_t by the monetary authority is thus not revealed to anyone--consumers or firms--until the transfers stop. If in equilibrium consumers spend all of the cash available to them as soon as they receive it, up to the point where $\alpha^{-1}U'(c)$ no longer exceeds the marginal price at which goods are available, then the information of producers and consumers will be the same at all stages: Each will know only the ex ante distribution of possible money supplies, and the quantity of money transferred and spent up to that point. In this case of a sequence of symmetric-information trading games, trading at each stage is equivalent to a Walrasian equilibrium. This is the interpretation developed in Eden (1994), in which there is a sequence of Walrasian spot markets, one after each new injection of cash.

There is thus a sense in which our results do not rely upon a non-Walrasian market structure, if the sequential revelation of information about the state of aggregate demand can be motivated in some other way. We find the sequential mechanism described here, however, to be an especially attractive model of the trading process. In Eden's model, it is fortuitous that the joint probability structure for the sequence of monetary injections within a period always implies that a larger total money supply corresponds to a longer sequence of injections, rather than to a larger quantity injected on each occasion. (In the latter case, the shock would be neutral.) In our model, the connection between a larger total money supply and a longer sequence of transactions at distinct prices in inevitable.

To sum up, in models in which goods are exchanged for money in a single centralized market, changes in money--anticipated or not--are neutral because all prices immediately adjust in proportion, leaving relative prices and goods quantities unaffected. If some of the affected prices do not respond in this way, either because they have been set in advance or because changing them involves costs, real effects can occur. One may think of the equilibrium studied in this paper as a formalization of the consequences of advance price setting on the part of sellers. In our setting, sellers lose nothing by pricing all units in advance, and if there were costs, however small, of maintaining flexibility, they would give it up. Our objective has been to show that, however important such commitments and costs may be in reality, they are in no sense <u>necessary</u> in generating real effects of monetary shocks.

We have given our producers up-to-the-minute information about all trading that has taken place anywhere in the system, and can easily interpret the equilibrium--as Eden does his--as one in which they have up-to-the-minute information on the supply of money in the system. We have given them the flexibility to make current pricing decisions, given their information, in a way that is entirely unconstrained by past actions. We have, however, compelled them to trade goods for cash, in a situation in which the terms on which they would wish to do so necessarily depend on monetary events they cannot know at the time. We have shown that, in this situation, they cannot avoid being led by monetary instability into inefficient production behavior of a kind that could not occur in an Arrow-Debreu economy. Moreover, in a context in which technology shocks do not occur, we have shown how an outside observer could easily misread their actions as efficient responses to such shocks.

Appendix: Proofs of Propositions 4.1 and 4.2.

For convenience, we restate the propositions in the text.

Proposition 4.1. In the maximization problem (4.3), suppose that

(i) the function $c(\theta)$ is continuous on Θ;

(ii) the function $p(z)$ is non-decreasing and right-continuous, with $p(z) > 0$ for all $0 \leq z \leq y$;

(iii) the function $\Pi(\theta)$ is continuous, and $\Pi(\theta) \geq 0$ for all $\theta \in \Theta$; and

(iv) the function $U(c)$ is bounded, continuous, strictly increasing, and concave, for all $c \geq 0$.

Then there exists a value function $v(m)$, defined for all $m \geq 1-\underline{\theta}$, that satisfies the Bellman equation (4.3a). Furthermore, $v(m)$ is bounded, continuous, strictly increasing, and concave.

Proof: The proof involves five parts. We first, (1), use the constraints (4.3b)-(4.3d) to express the decision variable c in (4.3a) in terms of $\tilde{m}$. Then, (2), we define an operator T associated with (4.3) and show that this operator takes the set of bounded continuous functions on $[1-\underline{\theta},\infty)$ into itself. We show, (3), that T has a unique fixed point v in this set of functions, the unique solution to (4.3). Then we show, (4), that v is increasing and, (5) that v is concave.

(1) For each $m \geq 1 - \underline{\theta}$, $\theta \in \Theta$, let $\Gamma(m,\theta)$ denote the interval of values for $\tilde{m}$ that satisfy (4.3b). Note that Γ is a continuous, compact-valued correspondence. Let D denote the graph of Γ, i.e., the set $(m,\theta,\tilde{m})$ satisfying the inequalities just mentioned. On the subset of D on which $\mu(m,\theta) > 0$ (i.e., on which $m > 1-\theta$) define the function

$$J(m,\theta,\tilde{m}) = \theta - \frac{\tilde{m} - \Pi(\theta) - \tau(\theta)}{\mu(m,\theta)} .$$

Note that $0 \le J(m,\theta,\tilde{m}) \le \theta$ by (4.3b). Likewise define

$$I(x,\theta) = \int_0^x \min[p(z),p(c(\theta))]dz$$

for arbitrary $x \ge 0$, $\theta \in \Theta$, using the convention that $p(z) = \infty$ for all $z \ge y$. By (i) and (ii), $I(x,\theta)$ is continuous and strictly increasing in x, with $I(0,\theta) = 0$ and $I(\infty,\theta) = \infty$. Hence the equation

$$\text{(A.1)} \qquad I(x,\theta) = J(m,\theta,\tilde{m})$$

has a unique solution $x(m,\theta,\tilde{m}) \ge 0$, and the unique solution to (4.3c) is given by

$$\text{(A.2)} \qquad c(m,\theta,\tilde{m}) = \mu(m,\theta)x(m,\theta,\tilde{m}) .$$

When $\mu(m,\theta) = 0$, let $c(m,\theta,\tilde{m}) = 0$. Then $c = c(m,\theta,\tilde{m})$ is the unique solution to (4.3c) and (4.3d) on D.

By (iii), J is continuous on the subset of D on which $\mu(m,\theta) > 0$, and by (i) and (ii), $I(x,\theta)$ is right continuous in θ. Hence on this subset of D, the function $c(m,\theta,\tilde{m})$ is continuous in $(m,\tilde{m})$ for each θ, and a right continuous function of θ for each $(m,\tilde{m})$. Morover, on this subset,

$$0 \le x(m,\theta,\tilde{m}) \le \frac{J(m,\theta,\tilde{m})}{p(\theta)} \le \frac{\theta}{p(0)},$$

which implies that $0 \le c(m,\theta,\tilde{m}) \le \mu(m,\theta)\theta/p(0)$. Since $p(0) > 0$ and Θ is bounded, it follows that $c(m,\theta,\tilde{m})$ has these continuity properties on all of D.

(2) Let $f(m)$ be any bounded, continuous function, defined for all $m \ge 1-\underline{\theta}$. Then

$$F(m,\theta,\tilde{m}) = U(c(m,\theta,\tilde{m})) + \beta f(\tilde{m})$$

is a bounded function on D. Furthermore, it follows from our results above that for any $\tilde{\theta} \ge \underline{\theta}$, there exists a right neighborhood N of $\tilde{\theta}$ such that F is continuous on D_N, the subset of D on which $\theta \in N$.

Now for any $m \ge 1-\underline{\theta}$, $\underline{\theta} \le \theta \le \bar{\theta}$, define

$$\phi(m,\theta) = \sup_{\tilde{m}\in\Gamma(m,\theta)} F(m,\theta,\tilde{m}) .$$

For any $\tilde{\theta} \ge \underline{\theta}$, let N be the right neighborhood of $\tilde{\theta}$ just referred to. Then by the theorem of the maximum (Stokey, Lucas, and Prescott, 1989, Theorem 3.6), $\phi(m,\theta)$ is continuous on the set $m \ge 1-\underline{\theta}$, $\theta \in N$. Thus we observe that $\phi(m,\theta)$ is a well-defined bounded function for all $m \ge 1-\underline{\theta}$, $\underline{\theta} \le \theta \le \bar{\theta}$; that for every $\theta \in \Theta$, $\phi(m,\theta)$ is a continuous function of m; and that for every $m \ge 1-\underline{\theta}$, $\phi(m,\theta)$ is a right-continuous function of θ.

Finally for any $m \ge 1-\underline{\theta}$, define

$$(Tf)(m) = \int \phi(m,\theta)\Phi(d\theta) . \tag{A.3}$$

Since $\phi(m,\theta)$ is a bounded, right-continuous function of θ, it is integrable, and the above expression is well-defined. It is also obviously bounded as a function of m. We wish to show that it is also continuous. Consider any sequence $\{m_n\}$ such that $m_n \geq 1-\underline{\theta}$ for each n, and $m_n \rightarrow m$. Since $\phi(m,\theta)$ is continuous in m for every θ, the functions $\{\phi(m_n,\cdot)\}$ converge pointwise to the function $\phi(m,\cdot)$. All of those functions are integrable, and they are uniformly bounded; hence, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int \phi(m_n,\theta)\Phi(d\theta) = \int \phi(m,\theta)\Phi(d\theta) .$$

Thus Tf is a continuous function of m.

We observe that (A.3) defines an operator T, mapping the set of bounded continuous functions on $[1-\underline{\theta},\infty)$ into itself. The existence of a function $v(m)$ satisfying (4.3a) then follows if we can show that there exists a fixed point of the mapping T.

(3) Let $\mathbf{F}$ denote the set of bounded continuous functions on $[1-\underline{\theta},+\infty)$, with the sup norm. For any functions $f,g,\in \mathbf{F}$, $f \geq g$ implies that $Tf \geq Tg$. For any function $f \in \mathbf{F}$ and any constant function a, $T(f+a) \leq TF + \beta a$. Thus the Blackwell conditions are satisfied (recall that $0 < \beta < 1$), and so T is a contraction (Stokey, Lucas, and Prescott, 1989, Theorem 3.3). Then since $\mathbf{F}$ is a complete metric space, T has a unique fixed point $v \in \mathbf{F}$. This function v necessarily satisfies (4.3a).

(4) We next show that $v(m)$ is monotonically increasing. Let $\mathbf{F}' \subset \mathbf{F}$ be the set of bounded, continuous functions that are also non-decreasing. We wish to show that T maps $\mathbf{F}'$ into itself as well.

Note that Γ is increasing in m, in the sense that $m' \geq m$ implies that $\Gamma(m,\theta) \subseteq \Gamma(m',\theta)$. Note also that $\mu(m,\theta)$ is strictly increasing as a function of m, as a result of which $J(m,\theta,\bar{m})$ is strictly increasing in m on the subset of D where $\mu(m,\theta) > 0$ (so that J is defined). Since $I(x,\theta)$ is strictly increasing in x, equation (A.1) defines a function $x(m,\theta,\bar{m})$ that is strictly increasing in m (on the domain of J). It follows that $c(m,\theta,\bar{m})$ is also strictly increasing in m on this domain. Consider now the unique point in D at which $\mu(m,\theta) = 0$, namely, $m = 1-\underline{\theta}$, $\theta = \underline{\theta}$, $\bar{m} = \Pi(\underline{\theta}) + 1 - \underline{\theta}$. If $(m,\theta,\bar{m})$ takes these values, and $m' > m$, then $\mu(m',\theta) > 0$, $J(m',\theta,\bar{m}) = \underline{\theta}$, $I(0,\theta) = 0$, so that $x(m',\theta,\bar{m}) > 0$ and $c(m',\theta,\bar{m}) > 0$. But $c(m,\theta,\bar{m}) = 0$, so that $c(m',\theta,\bar{m}) > c(m,\theta,\bar{m})$. Thus $c(m,\theta,\bar{m})$ is strictly increasing in m, on the entire domain D.

Now let f(m) be any function belonging to $\mathbf{F}'$. It follows that $F(m,\theta,\bar{m})$ is strictly increasing in m, on the domain D. Because Γ is increasing in m, it follows that $\phi(m,\theta)$ is strictly increasing in m, on the domain $m \geq 1-\underline{\theta}$, $\underline{\theta} \leq \theta \leq \bar{\theta}$. Finally, it follows from this that (Tf)(m) is strictly increasing in m, on the domain $m \geq 1-\underline{\theta}$. Thus $T:\mathbf{F}' \to \mathbf{F}'$, and furthermore $f \in \mathbf{F}'$ implies that Tf is strictly increasing. Since $\mathbf{F}'$ is a closed subset of the complete metric space $\mathbf{F}$, T must have a fixed point in $\mathbf{F}'$, which is the same function v(m) referred to above. Furthermore, since $v = Tv$, v(m) must be monotonically increasing in m, and not merely nondecreasing.

(5) Finally, we show that v(m) is concave. Let $\mathbf{F}'' \subset \mathbf{F}$ be the set of bounded, continuous functions that are also concave. We wish to show that T maps $\mathbf{F}''$ into itself as well.

We first show that the function $c(m,\theta,\bar{m})$, defined above, is concave in $(m,\bar{m})$, for any $\theta \in \Theta$. Let us define

$$K(\mu,\theta,\bar{m}) = \theta\mu + \Pi(\theta) + \tau(\theta) - \bar{m} .$$

Then (A.1) can equivalently be written

$$I(x,\theta) = \frac{K}{\mu}$$

which we can invert to obtain

$$x = \bar{x}(\frac{K}{\mu},\theta) .$$

Because $p(z)$ is non-decreasing, $I(x,\theta)$ is a convex function of x, for any θ. It follows that $\bar{x}(\frac{K}{\mu},\theta)$ is a concave function of $\frac{K}{\mu}$. Then

$$\bar{c}(K,\mu,\theta) = \mu\bar{x}(\frac{K}{\mu},\theta)$$

is a concave function of (K,μ), for any θ. But

$$c(m,\theta,\bar{m}) = \bar{c}[K(\mu(m,\theta),\theta,\bar{m}),\mu(m,\theta),\theta] .$$

Since $K(\mu,\theta,\bar{m})$ is linear in $(\mu,\bar{m})$, and $\mu(m,\theta)$ is linear in m, for any θ, it follows that $c(m,\theta,\bar{m})$ is concave in $(m,\bar{m})$.

Now let $f(m)$ be any function belonging to $\mathbf{F}''$. Then the concavity of $U(c)$ and $c(m,\theta,\bar{m})$ imply that $F(m,\theta,\bar{m})$ is concave in $(m,\bar{m})$, for any

$\theta \in \Theta$. Furthermore, for any $\theta \in \Theta$, the set of values $(m,\tilde{m})$ such that $(m,\theta,\tilde{m}) \in D$ is convex, from which it follows that $\phi(m,\theta)$ is concave in m. As this is true for all θ, the integral over θ must be concave in m as well, so that (Tf)(m) is concave in m.

Thus $T:\mathbf{F}'' \to \mathbf{F}''$. As $\mathbf{F}''$ is a closed subset of the complete metric space $\mathbf{F}$, T must have a fixed point in $\mathbf{F}''$, which must be the function v. Hence v(m) is concave in m. □

<u>Proposition 4.2</u>. Assume hypotheses (i)-(iv) of Proposition 4.1, and let $c(\theta) = c(1,\theta)$ solve the maximum problem (4.3) at $m = 1$. Suppose in addition that

(v) $c(\theta)$ is uniformly bounded away from zero on Θ;

(vi) the function U(c) is continuously differentiable for all $c > 0$, it is strictly concave (i.e., $U'(c)$ is monotonically decreasing), and $\lim_{c\to 0} U'(c) = +\infty$; and

(vii) $\Pi(\theta)$ is given by (4.4).

Then the value function v(m) that satisfies (4.3a) is differentiable at $m = 1$, and the derivative equals

$$v'(1) = (1-\beta)^{-1}\int \theta^{-1}U'(c(\theta))c(\theta)\Phi(d\theta) \ . \tag{4.7}$$

<u>Proof</u>. In order to show that v(m) is differentiable, we first consider the differentiability of another function, $\tilde{v}(m)$, that coincides with v(m)

at $m = 1$. For any $m_0 \geq 1-\underline{\theta}$, and any sequence of monetary injections $\theta = (\theta_t)$, let sequences $(\tilde{c}_t(m_0;\theta), \tilde{m}_t(m_0;\theta))$ be defined recursively by the relations

$$\tilde{m}_0(m_0;\theta) = m_0 \quad ,$$

$$\tilde{m}_{t+1}(m_0;\theta) = \mu(\tilde{m}_t(m_0;\theta),\theta_t)[1-\Pi(\theta_t)-\tau(\theta_t)] + \Pi(\theta_t) + \tau(\theta_t) \quad ,$$

$$\tilde{c}_t(m_0;\theta) = \mu(\tilde{m}_t(m_0;\theta),\theta_t)c(\theta_t) \quad .$$

For any m_0 and any sequence of monetary injections, the sequences $(\tilde{c}_t(m_0;\theta), \tilde{m}_t(m_0;\theta))$ represent a plan that satisfies (4.3b)- (4.3d) in all periods. Moreover, when $m_0 = 1$, this is the optimal plan, namely $\tilde{c}_t = c(\theta_t)$, $\tilde{m}_{t+1} = 1$ for all t. Then let $\tilde{v}(m_0)$ denote the level of utility obtained under this plan, i.e.,

$$\tilde{v}(m_0) = E(\sum_{t=0}^{\infty} \beta^t U(\tilde{c}_t(m_0;\theta)))$$

where the expectation is over the different possible histories of monetary injections θ. It is easily seen that for each t, $\tilde{c}_t(m_0;\theta)$ is a continuous function of θ, so that $U(\tilde{c}_t(m_0;\theta))$ is a bounded continuous function, with the same bounds for all t. Hence the integral involved in the above definition is well-defined for each t, and the uniform bounds imply that the infinite sum of integrals must converge. Hence $\tilde{v}(m_0)$ is well-defined for each $m_0 \geq 1-\underline{\theta}$.

We wish to show that $\tilde{v}(m_0)$ is differentiable at $m_0 = 1$. For

$\epsilon \geq -\underline{\theta}$, $\epsilon \neq 0$, let us define

$$\rho(\epsilon) = \frac{\tilde{v}(1+\epsilon) - \tilde{v}(1)}{\epsilon} .$$

Then we wish to show that the function $\rho(\epsilon)$ is continuous at $\epsilon = 0$; the limiting value $\rho(0)$ is then $\tilde{v}'(1)$. From the definition of the processes $(\tilde{c}_t, \tilde{m}_t)$, it is obvious that

$$\tilde{v}(m_0) = E(U(\mu(m_0,\theta_0)c(\theta_0))$$

$$+ \beta\tilde{v}(\mu(m_0,\theta_0)[1-\Pi(\theta_0)-\tau(\theta_0)] + \Pi(\theta_0) + \tau(\theta_0)))$$

where here the expectation is over the possible realizations of θ_0. This implies that for any $\epsilon \geq -\underline{\theta}$, $\epsilon \neq 0$,

$$\rho(\epsilon) = \int[u(\epsilon,\theta) + \beta\rho(\lambda(\theta)\epsilon)]\Phi(d\theta) \tag{A.4}$$

where

$$\lambda(\theta) = \frac{1-\Pi(\theta)-\tau(\theta)}{\theta} ,$$

$$u(\epsilon,\theta) = \frac{1}{\epsilon}[U(\frac{\epsilon+\theta}{\theta}c(\theta)) - U(c(\theta))] .$$

Note that (4.3b)-(4.3c), together with (4.4), imply that $0 \leq \Pi(\theta) \leq \theta$ for all θ, so that $0 \leq \lambda(\theta) \leq 1$ for all θ. And the facts that $U(c)$ is continuously differentiable for all $c > 0$, and that $c(\theta)$ is a continuous function bounded away from zero, imply that if we adjoin to the above definition the stipulation

$$u(0,\theta) = \frac{c(\theta)U'(c(\theta))}{\theta}$$

then $u(\epsilon,\theta)$ is a continuous function on the domain $\epsilon \geq -\underline{\theta}$, $\underline{\theta} \leq \theta \leq \bar{\theta}$. Finally the concavity of $U(c)$ implies that for any $\underline{\theta} \leq \theta \leq \bar{\theta}$,

$$0 \leq u(\epsilon,\theta) \leq \frac{U(c(\theta)) - U(0)}{\theta}$$

for all $\epsilon \geq -\underline{\theta}$. As the right hand expression is a bounded function of θ, it follows that $u(\epsilon,\theta)$ is a bounded function.

Now for any bounded continuous function $f(\epsilon)$ on the domain $\epsilon \geq -\underline{\theta}$, let

$$\text{(A.5)} \qquad (Tf)(\epsilon) = \int [u(\epsilon,\theta) + \beta f(\lambda(\theta)\epsilon)]\Phi(d\theta) \; .$$

From the properties just mentioned, it is evident that $u(\epsilon,\theta) + \beta f(\lambda(\theta)\epsilon)$ is well-defined for all $\epsilon \geq -\underline{\theta}$, $\theta \in \Theta$, and that it is furthermore a bounded continuous function on this domain. It then follows that $(Tf)(\theta)$ is well-defined for all $\epsilon \geq -\underline{\theta}$, and furthermore a bounded continuous function of ϵ on this domain. Thus (A.5) defines an operator T that maps the set of bounded continuous functions on the domain $[-\underline{\theta},\infty)$ into itself. As it satisfies the Blackwell conditions (with the sup norm), it is a contraction, and so has a unique fixed point. Comparison of (A.5) with (A.4) shows that the fixed point coincides with the function $\rho(\epsilon)$ defined earlier, for all $\epsilon \neq 0$. Since the fixed point is a continuous function, $\rho(\epsilon)$ is continuous at $\epsilon = 0$. Thus $\tilde{v}(m)$ is differentiable at $m = 1$, and the derivative is $\tilde{v}'(1) = \rho(0)$.

From (A.5) we observe that the fixed point, evaluated at $\epsilon = 0$, must satisfy

$$\rho(0) = \int [u(0,\theta) + \beta\rho(0)]\Phi(d\theta)$$

so that

$$\tilde{v}'(1) = (1-\beta)^{-1}\int \theta^{-1}U'(c(\theta))c(\theta)\Phi(d\theta) \quad \text{(A.6)}$$

Finally, since $\{\tilde{c}_t(m_0;\theta),\tilde{m}_t(m_0;\theta)\}$ represents a feasible plan, it follows that $\tilde{v}(m) \leq v(m)$ for all $m \geq 1-\underline{\theta}$. Furthermore, $\tilde{v}(1) = v(1)$. One also observes that $\tilde{c}_t(m_0;\theta)$ is a linear function of m_0, for each θ, as a result of which $\tilde{v}(m)$ is a concave function of m. Then the fact that $\tilde{v}(m)$ is differentiable at $m = 1$ implies that $v(m)$ is differentiable at that point as well, by the lemma of Benveniste and Scheinkman (Stokey, Lucas, and Prescott, 1989, Theorem 4.10), and that the derivative is $v'(1) = \tilde{v}'(1)$. Then (A.6) implies (4.7). □

Footnotes

[1] Rotemberg (1988) also stresses that the Prescott-Butters equilibrium need not involve any price commitments by sellers prior to the time at which an order is accepted.

[2] To be precise, $p(z)$ denotes the minimum element in the support of the distribution of prices at which goods are offered for sale after stage z is reached. This more precise definition is important in clarifying that an individual, atomistic producer cannot change the aggregate supply function $p(z)$ through his own pricing behavior.

[3] Here we assume that if buyers are indifferent between buying and not, they will always buy. This allows us to exclude the possibility of purchases ending at a stage z where $U'(z) = \alpha p(z)$ even though cash balances are not exhausted, and $U'(c) + \alpha[\theta - R(c)]$ is still <u>increasing</u> in c, for c in a right neighborhood of z. In fact, it is shown below that in equilibrium, $U'(z) > \alpha p(z)$ for all $z < c(\theta)$, so that buyers strictly prefer to buy at all stages prior to the terminal stage.

[4] This measurability assumption amounts to restricting attention to a particular type of equilibrium, in which it happens that producers' strategies vary with their index in a certain way. Consideration of equilibria of this particular form does not involve any assumption that producers coordinate their pricing decisions, any more than would a restriction to the much more special class of symmetric equilibria. Any symmetrical equilibrium is necessarily of this form. To simplify notation, we do not even formally define equilibrium except of this particular sort.

[5] Technically, we define this expression as the limit as $\epsilon \downarrow 0$ of the integral over the interval $[\epsilon, y]$. As $\Lambda(z)$ is a measurable function,

$\Lambda(F_j(z_j))$ is measurable for any right-continuous F_j. Furthermore, (3.1) imples that this function is non-negative, and bounded above by $\alpha^{-1}U'(\epsilon)$ on $[\epsilon,y]$. Hence $\Lambda(F_j(z_j))$ is integrable on any such interval. Moreover, the integral is non-decreasing as ϵ decreases, so the limit as $\epsilon\downarrow 0$ exists (though it may be infinite).

6 In the case of a measure Φ that contains atoms, this is a Prescott-Butters strategy only if the previous definition is relaxed to specify that for every $0 \le z_j < y$, <u>either</u> $F_j(z_j) = \inf\{z \mid p(z) \ge p_j(z_j)\}$, <u>or</u> $p(F_j(z_j)) = p_j(z_j)$. In other words, in the event that a positive fraction of aggregate sales occur at a single price p, it is not necessary that every seller j that intends to sell units at the price p sell <u>all</u> of them <u>as soon as</u> $p(z)$ reaches the level p; it is enough that all of j's goods with that price tag be sold by the time $p(z)$ rises <u>above</u> p.

Note that the equilibrium concept of Prescott and Butters is in fact an equilibrium in this broader class of strategies (for equilibria in which a positive fraction of all goods sell at a single price are possible in their framework). In the case that the weaker definition is required, the representation of the Prescott-Butters equilibrium is our formalism is somewhat awkward, in that our formalism requires seller j to specify at exactly <u>which</u> of the stages $\{z \mid p(z) = p\}$ his units will be sold, which appears inconsistent with the idea that j simply fixes the price tags before any sales occur and is thereafter completely passive. Nonetheless, it may be verified that the Prescott-Butters equilibrium corresponds exactly to an equilibrium in our sense, within the restricted class of strategies just defined.

7 Note that in this model, beyond a certain point a higher realization of the money supply has no effect on either prices or real activity. The

additional money is simply hoarded. This kind of "liquidity trap" is a typical feature of cash-in-advance models, and has nothing to do with the special market structure proposed here. See, for example, the discussion of the corresponding equilibrium with Walrasian spot markets in the two paragraphs below.

[8] Of course, real effects of anticipated money growth are possible in such models, if there are possibilities for substituting away from cash transactions. Such effects are entirely due to the fact that higher anticipated inflation increases the cost of using cash, and so increases the resort to inefficient alternatives. In this paper, we abstract entirely from this source of monetary non-neutrality.

List of References

Butters, Gerard. 1977. "Equilibrium Distributions of Sales and Advertising Prices." Review of Economic Studies 44:467-491.

Eden, Benjamin. 1990. "Marginal Cost Pricing when Spot Markets are Complete." Journal of Political Economy 98:1293-1306.

__________ 1994. "The Adjustment of Prices to Monetary Shocks when Trade is Uncertain and Sequential." Journal of Political Economy, forthcoming.

__________ and Zvi Griliches. 1993. "Productivity, Market Power, and Capacity Utilization When Spot Markets are Complete." American Economic Review 83, Papers and Proceedings:219-223.

Friedman, Milton, and Anna J. Schwartz. 1963. A Monetary History of the United States, 1867-1960. Princeton: Princeton University Press.

Lucas, Robert E., Jr. 1972. "Expectations and the Neutrality of Money." Journal of Economic Theory 4:103-124.

Prescott, Edward C. 1975. "Efficiency of the Natural Rate." Journal of Political Economy. 83: 1229-1236.

Rotemberg, Julio J. 1988. "Rationing in Centrally Planned Economies." M.I.T. working paper.

__________ and Lawrence H. Summers. 1990. "Inflexible Prices and Labor Hoarding's Role in Explaining Procycical Productivity." Quarterly Journal of Economics 105:851-874.

Sargent, Thomas J. 1987. Dynamic Macroeconomic Theory. Cambridge: Harvard University Press.

Stokey, Nancy L., Robert E. Lucas, Jr., and Edward C. Prescott. 1989. Recursive Methods in Economic Dynamics. Cambridge: Harvard University Press.