CONDENSED MATHEMATICS AND APPLICATIONS: SUMMER SCHOOL ON ARITHMETIC AND p-ADIC GEOMETRY IN CHILE

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Contents

Condensed mathematics is a recent theory developed by Clausen and Scholze [CS19] whose main motivation is to give an elegant and practical solution to the classical problem of properly mixing algebra, analysis and topology. On the other hand, Analytic Geometry [CS20, CS24] is a generalization of algebraic geometry within the realm of condensed mathematics that aims to organize in a single package essentially all other geometric theories such as topological and smooth manifolds, complex analytic spaces, rigid analytic spaces, formal schemes, p-adic manifolds, adic and perfectoid spaces, Berkovich spaces, etc.

In this mini course we will review the basics of the theory of condensed mathematics and will explain one of its applications in algebraic and analytic geometry: Serre duality. We will highlight the importance of the use of the condensed formalism to even being able of make basic "natural constructions" that otherwise are difficult to formalize. Another very important tool used in this theory are abstract six functor formalisms (eg. [HM24]) which shall be mention across the course.

References for the interested reader. We list some of the existing references of condensed mathematics and analytic geometry:

(1) The first written reference are the lecture notes of condensed mathematics [CS19] where the foundations of condensed mathematics were presented, as well as the construction of solid abelian groups.
(2) In the lecture notes [CS20] the category of liquid modules are constructed (a very difficult theorem!), theory that provides a completed tensor product in the real number. Some first aspects of analytic geometry such as analytic rings are also introduced.
(3) [Man22b] develops the proper higher categorical foundations for condensed mathematics and analytic rings. Mann also introduced a notion of descendability for analytic rings which is used

to prove descent in analytic geometry. He applies this theory to construct a six functor formalism in torsion solid quasi-coherent sheaves on v-stacks. In [Man22a] Mann constructs a six functor formalism for nuclear ℓ-adic sheaves on v-stacks.

(4) In [And21] Andreychev proved descent for solid quasi-coherent, nuclear and perfect sheaves on analytic adic spaces.

(5) In [Mik22] Mikami proves fppf descent for categories of solid quasi-coherent sheaves on schemes and rigid spaces.

(6) In the course [CS22] CLausen and Scholze revisit the foundations of of complex analytic geometry from the point of view of condensed mathematics via the liquid tensor product. In particular, they define a category of (liquid) quasi-coherent sheaves on complex analytic manifolds, prove Serre duality, and Grothendieck-Riemann-Roch.

(7) The lecture notes of Scholze [Sch23] and the paper [HM24] of Heyer-Mann provide the tools from the theory of abstract six functor formalisms used in the theory of analytic stacks. In addition, Heyer-Mann construct a six functor formalism for abelian sheaves on condensed anima, and apply this technology to smooth representations of profinite groups.

(8) In [RJRC22, RJRC23] in joint with Rodrigues Jacinto we use the theory of solid abelian groups to define and study the category of solid locally analytic representations of p-adic Lie groups.

(9) The course on analytic stacks [CS24] presents the new foundations of analytic geometry via light condensed mathematics. It also introduces the category of analytic stacks and give very important examples. In [RC24b] you can also find some written notes (still in construction) summarizing some aspects of the course on Analytic Stacks.

(10) In [RC24a] we introduce the category of derived adic spaces and the analytic de Rham stack in rigid geometry. This last object provides a geometric definition of a category of analytic D-modules in rigid geometry.

1. Quick introduction to condensed mathematics

1.1. **Condensed sets.** From Grothendieck's point of view, given a category $\mathcal{C}$ (eg. abelian groups, topological spaces, complex manifolds, etc.), one should study objects in $\mathcal{C}$ by studying morphisms between objects instead. For instance, let Top be the category of topological spaces. What Grothendieck says is that instead of study a topological space T on its own one should study, for example, continuous maps $S \to T$ for objects $S \in \mathsf{Top}$.

Suppose that we are willing to subtract a "good class" of topological spaces which are not pathological from the point of view of analysis, namely, spaces that are in a broad sense determined by "sequences". Examples of these include Banach spaces, Fréchet spaces, CW complexes, metrizable compact Hausdorff spaces, etc. How to restrict ourselves to such a class of topological spaces? A first step involves in choosing which spaces will be the "building blocks" of any other space. Experience obtained from decades suggest that a reasonable class is that of "compact Hausdorff spaces". Indeed, a topological space T is called *compactly generated* if there is a family of compact Hausdorff spaces $\{X_i\}$ and a continuous surjective map $\bigsqcup_i X_i \to T$ such that T has the quotient topology. Essentially any space appearing in classical geometry and analysis is compactly generated (even generated by *metrizable* compact Hausdorff spaces).

Let $\mathsf{CHaus} \subset \mathsf{Top}$ be the full subcategory of compact Hausdorff spaces. Before understanding how a general space can be rebuilt from objects in CHaus we need to understand how this category behaves, in particular how to rebuilt compact Hausdorff spaces from suitable maps. This is explained in the following proposition:

Proposition 1.1.1. *Let $f : S \to T$ be a surjective map of compact Hausdorff spaces. Then f is a quotient map. In particular, f is an homeomorphism if and only if it is a bijection.*

Exercise 1.1.2. Prove Proposition 1.1.1.

The previous proposition says that it is really easy to get new compact Hausdorff spaces from a given compact Hausdorff space: one only needs to define a closed equivalence relation. From now on we shall endow CHaus with the Grothendieck topology generated by finite disjoint unions and continuous surjective mas. It is then natural to ask for a class of compact Hausdorff spaces that are "as simple as possible" from an algebraic point of view, i.e. for a rather explicit base for the Grothendieck topology of CHAus. This class of topological spaces are precisely the profinite sets:

Definition 1.1.3. A profinite set is compact Hausdorff space S which is totally disconnected, i.e. such that the only connected components of S are its points. We let $\mathsf{Prof} \subset \mathsf{CHaus}$ be the full subcategory of compact Hausdorff spaces.

There are equivalent ways to characterize profinite sets, some more algebraic than others.

Lemma 1.1.4. *The following categories are equivalent:*

(1) *The category of profinite sets* Prof.
(2) *The pro-category of finite sets* $\mathrm{Pro}(\mathsf{Fin})$*, namely, the category whose objects are inverse systems of finite sets* $(S_i)_{i\in I}$ *and maps given by*
$$\mathrm{Map}_{\mathrm{Pro}(\mathsf{Fin})}((S_i)_{i\in I},(T_j)_{j\in J}) = \varprojlim_{j\in J}\varinjlim_{i\in I}\mathrm{Map}_{\mathsf{Fin}}(S_i,T_j).$$
(3) *The opposite category of boolean rings.*

The functor from (2) to (1) goes by sending an inverse system $(S_i)_{i\in I}$ *to the limit* $S=\varprojlim_i S_i$ *endowed with the limit topology. The functor from (1) to (2) goes by taking a totally disconnected compact Hausdorff space* S *and mapping it to the inverse system* $(S_i)_{i\in I}$ *where* $S\to S_i$ *runs over all the surjective maps from* S *to finite sets. The functor from (3) to (1) goes by mapping a Boolean algebra* A *to its Zariski spectrum* $S=\mathrm{Spec}A$*. The functor from (1) to (3) maps a totally disconnected compact Hausdorff space* S *to the space of continuous functors* $A=C(S,\mathbb{F}_2)$ *with values in the field of two elements.*

Exercise 1.1.5. Prove Lemma 1.1.4.

Exercise 1.1.6. By Urysohn's metrization theory a metrizable compact Hausdorff space is the same as a 2-countable compact Hausdorff space, i.e. a compact Hausdorff space with a countable basis of open neighborhoods. We say that a profinite set is *light* if it is metrizable, let $\mathsf{Prof}^{\mathrm{light}} \subset \mathsf{Prof}$ be the full subcategory of light profinite sets. Following Lemma 1.1.4 show that the following categories are equivalent:

(1) The category of light profinite sets $\mathsf{Prof}^{\mathrm{light}}$.
(2) The countable pro-category of finite sets $\mathrm{Pro}_{\mathbb{N}}(\mathsf{Fin}) \subset \mathrm{Pro}(\mathsf{Fin})$, i.e. the full subcategory generated by sequential inverse systems.
(3) The opposite category of countable boolean rings.

Moreover, show that any light profinite set is a quotient of the Cantor set $C=\prod_{\mathbb{N}}\{0,1\}$,

Proposition 1.1.7. *Let* X *be a compact Hausdorff space. Then there is a profinite set* S *and a surjective map* $S\to X$.

Sketch of the proof. Let X be a compact Hausdorff space and let X^{δ} be the underlying set of X endowed with the discrete topology. The map $X^{\delta}\to X$ naturally extends to the Stone Čech compactification $f:\beta(X^{\delta})\to X$. The map f is clearly surjective and $\beta(X^{\delta})$ is a profinite set. □

Example 1.1.8. Let $I=[0,1]$ be the compact interval in $\mathbb{R}$. From classical analysis we know that any number $a\in I$ has a decimal expansion $a=0.a_1a_2a_3\cdots$ with $a_i\in\{0,\ldots,9\}$ which can be equally written as the series $a=\sum_{i=1}^{\infty}\frac{a_i}{10^i}$. Conversely, any such a sequence of numbers $(a_n)_n$ gives rise an element in $[0,1]$. Consider $S=\prod_{\mathbb{N}_{\geq 1}}\{0,\cdots,9\}$, then the map sending $(a_n)_{n\geq 1}$ to $a=\sum_{i+1}^{\infty}\frac{a_i}{10^i}$ defines a continuous surjective map $S\to[0,1]$ which is then a quotient map of compact Hausdorff spaces. In other words, the classical decimal expansions of real numbers are just a way to write $[0,1]$ as a quotient of a profinite set!

Exercise 1.1.9. Let X be a metrizable compact Hausdorff space, show that there exists a metrizable profinite set S and a surjective map $S \to X$. Hint: consider finite closed covers of X refined by open covers.

By Proposition 1.1.7, we have a good candidate to our goal of studying "nice" topological spaces from some concrete "building blocks", namely, use profinite sets as basic objects. This leads to the starting point of condensed mathematics, namely, condensed sets:

Definition 1.1.10. A condensed set T is a sheaf on Prof with values in sets. That is, it is a functor

$$T : \mathsf{Prof}^{\mathrm{op}} \to \mathsf{Set}$$

such that

(i) One has $T(\emptyset) = *$.
(ii) For S, S' profinite sets $T(S \sqcup S') = T(S) \times T(S')$.
(iii) For a surjective map of profinite sets $S' \to S$ we the following diagram is an equalizer

$$T(S) \to T(S') \rightrightarrows T(S' \times_S S')$$

where the double arrow correspond to the two projections from $S' \times_S S'$ to S.

We let $\mathrm{Cond}(\mathsf{Set})$ be the category of condensed sets, one defines in a similar way condensed abelian groups, condensed rings, etc.

Remark 1.1.11. In Definition 1.1.10 one could have consider compact Hausdorff spaces instead of profinite sets. The advantage of using profinite sets is that they are rather simple and can be built up purely from set theory without appealing to topology!

Remark 1.1.12. Definition 1.1.10 is not the official definition of condensed sets as they lack of the set theoretic condition of [CS19]. In the current formulation of condensed mathematics we prefer to use only *light profinite sets* in the definition of condensed sets (also called *light condensed sets*). There are several reasons for that choice as it is explained in [CS20], but one of the main justifications is that any space or topological ring that appears in nature in analytic, algebraic or arithmetic geometry has topology determined by light profinite sets. We will denote $\mathrm{Cond}(\mathsf{Set})^{\mathrm{light}} \subset \mathrm{Cond}(\mathsf{Set})$ for the full subcategory of light condensed sets (this is equivalent to the full subcategory generated under colimits by light profinite sets).

The category $\mathrm{Cond}(\mathsf{Set})$ is a topos, i.e. the category of sheaves on a site. This makes condensed sets a really convenient framework to perform algebraic constructions that behave as expected, eg. the category of abelian group objects in $\mathrm{Cond}(\mathsf{Set})$ (equivalently, $\mathrm{Cond}(\mathsf{Ab})$) is a Grothendieck abelian category (after fixing the size of the profinite sets by a suitable cardinal, eg. by taking light profinite sets) satisfying many of Grothendieck axioms [CS19, Theorem 2.2].

The promised relation between topological spaces and condensed sets is summarized in the following construction:

Definition 1.1.13. Let Top_ω be the category of compactly generated topological spaces. We define the condensification functor $\underline{(-)} : \mathsf{Top}_\omega \to \mathrm{Cond}(\mathsf{Set})$ mapping a topological space X to the sheaf on profinite sets given by $S \mapsto \underline{X}(S)$ for $S \in \mathsf{Prof}$.

Remark 1.1.14. For the construction of Definition 1.1.13 to give a condensed set it is necessary for the topological space T to be $T1$; this is due to the fact that the definition of condensed set involves an accessibility condition on the functor which would fail otherwise, see [CS19, Warning 2.14]. If one restricts to light condensed sets then this construction always produces a light condensed set thought one losses some information in the process. In practice all condensed sets we will care are light an all topological spaces for which we will like to construct its condensifiation are T1.

By Yoneda's embedding, the condensification functor induces a fully faithful embedding $\mathsf{Prof} \to \mathrm{Cond}(\mathsf{Set})$. A condensed set T is said quasi-compact if there is an epimorphism $\underline{S} \to T$ from a profinite set S. More generally, a map of condensed sets $T' \to T$ is said quasi-compact if for all profinite set S and any map $\underline{S} \to T$ the fiber product $T' \times_T S$ is quasi-compact. Finally, a map of condensed sets $T' \to T$ is said quasi-separated if the diagonal is quasi-compact, and a condensed set T is quasi-separated if $T \to *$ is quasi-separated. The following theorem explains the relationship between topological spaces and condensed sets:

Theorem 1.1.15 ([CS20, Propositions 1.2 Proposition 4.13])**.** *Let Top_ω be the category of compactly generated topological spaces. The following hold:*

(1) *The condensification functor of (1.1.13) is fully faithful when restricted to Top_ω.*
(2) *The condensification functor has a left adjoint mapping a condensed set T to the topological space $T(*)_{\mathrm{top}} \in \mathsf{Top}_\omega$ whose underlying set is $T(*)$ and with topology generated by all the maps*

$$S = \underline{S}(*) \to T(*)$$

induced by maps $\underline{S} \to T$ of condensed sets with $S \in \mathsf{Prof}$.
(3) *The functor $\underline{(-)}$ induces an equivalence of categories*

$$\mathsf{CHauss} \xrightarrow{\sim} \mathrm{Cond}(\mathsf{Set})^{\mathrm{qcqs}}$$

between compact Hausdorff spaces and qcqs condensed sets.
(4) *The functor $\underline{(-)}$ induces an equivalence of categories*

$$\mathrm{Ind}_{\mathrm{inj}}(\mathsf{CHauss}) \xrightarrow{\sim} \mathrm{Cond}(\mathsf{Set})^{qs}$$

between the ind-category of compact Hausdorff spaces along injective maps and the category of quasi-separated condensed sets. Moreover, if $X = \varinjlim_n X_n$ is a topological space given as a sequential colimit of compact Hausdorff spaces by injective transition maps, then the natural map

$$\varinjlim_n \underline{X}_n \xrightarrow{\sim} \underline{X}$$

is an isomorphism of condensed sets.
(5) *Let T be a quasi-separated condensed set, then quasi-compact injections $Z \subset T$ are in bijection with closed subspaces of $T(*)_{\mathrm{top}}$ via $Z \mapsto Z(*)_{\mathrm{top}}$.*

Let us disaggregate the previous theorem.

i. Part (1) says that categorically speaking we are not loosing any information for *good* topological spaces, quite the opposite, we are enriching the theory of topological spaces to condensed sets.
ii. Part (2) says that, nonetheless, a condensed set has an underlying topological space. Of course, by fully faithfulness, given X a compactly generated topological space the natural map $\underline{X}(*)_{\mathrm{top}} \to X$ is an homeomorphism.
iii. Part (3) says that the *compact objects* in both categories are identified; where in the topological side we literally have compact spaces and in the condensed side we use its categorical counterpart, namely qcqs objects.
iv. Part (4) described *Hausdorff objects* in the condensed side (i.e. quasi-separated spaces) in terms of a very precise construction involving compact Hausdorff spaces. Moreover, it says that the condensification functor commutes with countable colimits of compact Hausdorff spaces along injective maps.
v. Finally, part (5) says that the categorical analogues of closed subspaces in a quasi-separated condensed set are in bijection with usual closed subspaces of the underlying topological space.

Exercise 1.1.16. Read (or do!) the proof of Theorem 1.1.15.

A special class of condensed sets are of course those arising from discrete topological spaces. We say that a condensed set is *discrete* if it is of this form. Equivalently, discrete condensed sets are just the essential image of the fully faithful embedding

$$\mathsf{Set} \to \mathrm{Cond}(\mathsf{Set}).$$

1.2. **Condensed abelian groups.** As it was mentioned above the category of condensed abelian groups $\mathrm{Cond}(\mathsf{Ab})$ is a very nice abelian category (for a precise description of nice see [CS19, Theorem 2.2]). The category $\mathrm{Cond}(\mathsf{Ab})$ has a natural tensor product (as any category of abelian objects in a topos do): let N, M be condensed abelian groups, them $N \otimes M$ is the sheafification of the presheaf mapping a profinite set S to $N(S) \otimes M(S)$.

Exercise 1.2.1. Let N, M be condensed abelian groups, show that $(M \otimes N)(*) = M(*) \otimes N(*)$. This can be interpreted as saying that the tensor product in $\mathrm{Cond}(\mathsf{Ab})$ is still algebraic even though it produces a non-discrete condensed set, for instance $\underline{\mathbb{R}} \otimes \underline{\mathbb{R}}$ is a non-discrete condensed structure on the algebraic tensor product $\mathbb{R} \otimes \mathbb{R}$ which is huge! Later in Section 3 we will see how to define and construct non-trivial *completed tensor products* which are the first step to the theory of Analytic Geometry.

We highlight some further properties of condensed abelian groups:

(1) As it holds in any topos, the forgetful functor $\mathrm{Cond}(\mathsf{Ab}) \to \mathrm{Cond}(\mathsf{Set})$ has a left adjoint given by the *free condensed abelian group*. Explicitly, given T a condensed set its free condensed abelian group is the sheafification of the presehaf on Prof mapping S to

$$S \mapsto \mathbb{Z}[T(S)]$$

where the right hand side term is the free group generated by the set $T(S)$.

(2) The tensor product in $\mathrm{Cond}(\mathsf{Ab})$ has a right adjoint given by an internal $\underline{\mathrm{Hom}}$ space. Concretely, given N, M condensed abelian groups and $S \in \mathsf{Prof}$ one defines the internal Hom space by the formula

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(N, M)(S) = \mathrm{Hom}_{\mathbb{Z}}(N \otimes \mathbb{Z}[S], M).$$

(3) Let us mention a very important difference between condensed and light condensed abelian group. Let $\mathrm{Cond}(\mathsf{Ab})$ be the category of abelian groups, since the extremally totally disconnected profinite sets S are projective objects in the category of profinite sets, $\mathbb{Z}[S]$ is a compact projective condensed abelian group. Therefore, the category of condensed abelian groups is compactly projective generated. Unfortunately products of extremally totally disconnected spaces is not extremally totally disconnected (unless one is finite), and $\mathbb{Z}[S]$ is **not internally projective** as condensed abelian group [CS20, Proposition 4.8].

On the other hand, let $\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}}$ be the full subcategory of light condensed abelian groups. It is never true that light profinite sets are totally disconnected unless they are finite, it is also unlikely that $\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}}$ has enough compact projective modules. However, there is a very important compact projective object given by $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ which **is internally projective** [RC24b, Theorem 2.3.3].

It is convenient to have a mental picture of how some condensed abelian groups look like. The following proposition provides an explicit description for the free condensed abelian groups on profinite sets:

Proposition 1.2.2 ([CS20, Proposition 2.1])**.** *Let $S = \varprojlim_i S_i$ be a profinite set, written as an inverse limit of finite sets S_i. For any n, let $\mathbb{Z}[S_i]_{\ell^1 \leq n}$ be the finite set of sums $\sum_{s \in S_i} a_s[s]$ such that $\sum_i |a_s| \leq n$. Then there is a natural isomorphism of condensed abelian groups*

$$\mathbb{Z}[S] \cong \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\ell^1 \leq n} \subset \varprojlim_i \mathbb{Z}[S_i].$$

Exercise 1.2.3. Read the proof of Proposition 1.2.2.

An important feature of the condensification functor of Definition 1.1.13 is that it preserves topological abelian groups, topological rings, and topological modules over topological rings. Indeed, the functor $X \mapsto \underline{X}$ preserves limits, in particular cartesian products. Any of the previous algebraic structures in sets are given by the datum of certain diagrams involving finitely many products of a set (in more sophisticated terms we say that $\underline{(-)}_X$ is symmetric monoidal for the Cartesian product structure and so it preserves algebras over operads). For instance, the datum of a topological abelian group consists on a topological space M together with continuous maps $+ : M \times M \to M$ (the addition), $- : M \to M$ (inverses), and $0 : * \to M$ (the zero or unit), satisfying some commutative diagrams.

It is typical in real analysis that the Hom space between two topological real groups (eg. Banach spaces) has different topologies that one can attach. Even though those topologies are important for different reasons, condensed mathematics tells you what the canonical topology is from a categorical point of view:

Lemma 1.2.4. *Let A and B be Hausdorff topological groups and suppose that A is compactly generated. Let $\mathrm{Hom}'(A, B)$ be the topological abelian group defined by the compact open topology. Then there is a natural isomorphism of condensed abelian groups*

$$\underline{\mathrm{Hom}'(A, B)} \xrightarrow{\sim} \underline{\mathrm{Hom}}(\underline{A}, \underline{B}).$$

Exercise 1.2.5. Let us prove Lemma 1.2.4.

(1) Let T, T' be condensed sets, show that the mapping space $\mathrm{Map}(T, T')$ between condensed sets has a natural enrichment to a condensed set denoted by $\underline{\mathrm{Map}}(T, T')$.

(2) Let X and Y be topological Hausdorff spaces with X compactly generated, and let $\mathrm{Map}(X, Y)$ be the topological space of continuous maps endowed with the compact open topology. Show that there is a natural continuous map

$$\mathrm{Map}(X, Y) \times X \to Y$$

which gives rise to a map of condensed sets

$$\underline{\mathrm{Map}'(X, Y)} \to \underline{\mathrm{Map}}(\underline{X}, \underline{Y}).$$

(3) We now construct a map $\underline{\mathrm{Map}}(\underline{X}, \underline{Y}) \to \underline{\mathrm{Map}'(X, Y)}$. Let $S \in \mathrm{Prof}$ and let $f \in \underline{\mathrm{Map}}(\underline{X}, \underline{Y})$. The datum of f is equivalent to a map of condensed sets $\underline{S} \times \underline{X} \to \underline{Y}$ which is the same as a continuous map of topological spaces $F : S \times X \to Y$. Show that F induces a continuous map

$$S \to \mathrm{Map}'(X, Y)$$

and so an element in $\underline{\mathrm{Map}'(X, Y)}(S)$. This produces the desired map.

(4) Show that the maps of (2) and (3) are inverses. Deduce that

$$\underline{\mathrm{Map}'(X, Y)} = \underline{\mathrm{Map}}(\underline{X}, \underline{Y}).$$

(5) Obtain Lemma 1.2.4 from (4) by writing $\underline{\mathrm{Hom}}(A, B)$ as a suitable equalizer involving condensed mapping spaces.

1.3. **Cohomology.** A last very important feature that we will briefly mention is the fact that condensed mathematics recovers sheaf cohomology on topological spaces. This is proven in [CS19, Theorem 3.2]. In this lecture we will explain (but not prove) a more powerful and general statement about derived categories of abelian sheaves. We first require an easy lemma

Lemma 1.3.1. *Let S be a profinite set, then there is a natural equivalence of abelian categories*

$$\mathrm{Sh}(S, \mathbb{Z}) \cong \mathrm{Mod}(C(S, \mathbb{Z}))$$

between abelian sheaves on S and modules over the locally constant functions on S with values in $\mathbb{Z}$. The functor sends a sheaf $\mathcal{F}$ over S to $\mathcal{F}(S)$, and a $C(S, M)$-module M to the sheaf mapping an open-compact subspace $U \subset S$ to $C(U, \mathbb{Z}) \otimes_{C(S,M)} M$.

Exercise 1.3.2. Prove the Lemma 1.3.1.

As a corollary of Lemma 1.3.1 we deduce that the ∞-derived category of abelian sheaves on a profinite set S is naturally equivalent to the derived category of modules over $C(S,\mathbb{Z})$. We then deduce that derived category $D(S,\mathbb{Z})$ of abelian sheaves on profinite sets satisfy hyper-descent:

Proposition 1.3.3. *Let $S_\bullet \to S$ be an hypercover of profinite sets. Then the natural map*

$$D(S,\mathbb{Z}) \xrightarrow{\sim} \varprojlim_{[n]\in\Delta} D(S_n,\mathbb{Z})$$

is an equivalence.

Exercise 1.3.4. Show that for any map of profinite sets $S' \to S$ the map of rings $C(S,\mathbb{Z}) \to C(S',\mathbb{Z})$ is flat. Use this fact to show Proposition 1.3.3.

Having proven hyperdecent for the ∞-derived categories of abelian sheaves on profinite sets we can descend their formation to condensed sets (even condensed anima!)

Definition 1.3.5. Let T be a condensed set, we define its ∞-derived category of abelian complexes as

$$D(T,\mathbb{Z}) = \varprojlim_{S\to T} D(S,\mathbb{Z})$$

where $S \to T$ runs over all the maps from profinite sets to T.

Exercise 1.3.6.
(1) Show that the categories $D(T,\mathbb{Z})$ for T a condensed set have a natural t-structure such that for any map $f: T' \to T$ the pullback $f^*: D(T,\mathbb{Z}) \to D(T',\mathbb{Z})$ is t-exact. Hint: first prove the claim for prifinite sets.
(2) Deduce that $D(T,\mathbb{Z})$ has well defined left bounded and right bounded subcategories $D^+(T,\mathbb{Z})$ and $D^-(T,\mathbb{Z})$ respectively.

The following result follows from the hyper-descent property of Proposition 1.3.3.

Corollary 1.3.7. *Let $X_\bullet \to T$ be an hypercover of T by terms given by disjoint unions of profinite sets, then the natural map*

$$D(T,\mathbb{Z}) \to \varprojlim_{[n]\in\Delta} D(X_n,\mathbb{Z})$$

is an equivalence.

The following theorem compares the derived categories of condensed sets with the derived categories of some classical topological spaces.

Theorem 1.3.8 ([Sch23, Proposition 7.3])**.** *Let X be a locally compact Hausdorff space and let $\underline{X}$ be its condensification. Then there is a natural equivalence of ∞-categories*

$$D(\underline{X},\mathbb{Z}) \cong \widehat{D}_{\mathrm{top}}(X,\mathbb{Z})$$

between the derived category of the condensed set $\underline{X}$ and the left completion of the derived category of the topological space X.

Proposition 1.3.9. *Let X be a locally Hausdorff space locally of finite dimension, i.e., locally on X it admits a closed embedding into an open of $\mathbb{R}^n$. Then the derived category of sheaves $D_{\mathrm{top}}(X)$ is already left complete and so we have an equivalence of ∞-categories*

$$D(\underline{X},\mathbb{Z}) \cong D_{\mathrm{top}}(X,\mathbb{Z}).$$

Exercise 1.3.10. In this exercise we prove Theorem 1.3.8 and Proposition 1.3.9. Let X be a locally compact Hausdorff space.

(1) Let X_{cl} be the site given by compact subspaces of X with covers given by finitely many jointly surjective maps $\{Z_i \to Z\}_{i=1}^n$ which can be refined by open covers. Show that there is a natural equivalence between sheaves for the site X_{cl} and sheaves on the topological space X.

(2) Let X_{Cond} be the site consisting on maps $C \to X$ with C compact Hausdorff space and with coves given by jointly surjecive maps. Show that there is a morphism of sites

$$f : X_{\mathrm{Cond}} \to X_{\mathrm{cl}}.$$

(3) Prove that the morphism of sites f gives rise to fully faithful embedding of left bounded derived categories of abelian sheaves

$$f^* : D^+_{\mathrm{top}}(X, \mathbb{Z}) \cong D^+(X_{\mathrm{cl}}, \mathbb{Z}) \hookrightarrow D^+(X_{\mathrm{Cond}}, \mathbb{Z}).$$

(4) Construct a fully faithful embedding

$$D^+(\underline{X}, \mathbb{Z}) \hookrightarrow D^+(X_{\mathrm{Cond}}, \mathbb{Z}).$$

Hint: first do it for profinite sets, then use descend in the site X_{Cond}.

(5) Show that as subcategories of $D(X_{\mathrm{Cond}}, \mathbb{Z})$ one has

$$D^+(\underline{X}, \mathbb{Z}) \cong D^+_{\mathrm{top}}(X, \mathbb{Z}).$$

By taking left completions deduce that there is a natural equivalence

$$D(\underline{X}, \mathbb{Z}) \cong \widehat{D}_{\mathrm{top}}(X, \mathbb{Z}).$$

(6) Let A be an abelian group, deduce that one has a natural equivalence of objects in $D(\mathbb{Z})$

$$R\Gamma(\underline{X}, A) \cong R\Gamma_{\mathrm{Sheaf}}(X, A)$$

where the left hand side is the condensed cohomology of the condensed set $\underline{X}$, and the right hand side is sheaf cohomology.

(7) Suppose that X is locally of finite cohomological dimension, show that $D_{\mathrm{top}}(X, \mathbb{Z})$ is left complete, that its, that for all $F \in D_{\mathrm{top}}(X)$ the natural map

$$F \to \varprojlim_n \tau^{\geq -n} F$$

is an equivalence.

Previously we discussed condensed cohomology of discrete sheaves. The following is an important condensed cohomology computation for a non-discrete condensed abelian group.

Theorem 1.3.11 ([CS19, Theorem 3.3])**.** *Let S be any compact Hausdorff space, then we have a natural equivalence*

$$R\Gamma_{\mathrm{Cond}}(S, \mathbb{R}) = C(S, \mathbb{R}).$$

2. Solid abelian groups, solid quasi-coherent sheaves and Serre duality

In the previous talk we introduced condensed sets and condensed abelian groups, we explained how (nice) topological spaces embed into condensed sets, and that for the sheaf cohomology of locally compact Hausdorff spaces, working with the topological space or condensed set there is no virtual difference.

However, another of the desires in analytic geometry is to have suitable notions of completed tensor products that: 1) capture the already ad hoc completed tensor products appearing in nature, 2) has a purely algebraic nature so that one can apply methods of homological algebra (or more generally higher algebra). In this talk we are going to discuss one of the completed tensor products that naturally shows up in non-archimedean and algebraic geometry, namely, the *solid tensor product*.

In order to give the most recent approximation to this theory we will restrict our set up of condensed mathematics to light profinite sets $\mathsf{Prof}^{\mathrm{light}}$ and light condensed sets $\mathrm{Cond}(\mathsf{Set})^{\mathrm{light}}$. More concretely, $\mathsf{Prof}^{\mathrm{light}}$ consists on metrizable totally disconnected compact Hausdorff spaces (eq. countable limits of finite sets), and $\mathrm{Cond}(\mathsf{Set})^{\mathrm{light}}$ is the category of sheaves in sets on $\mathsf{Prof}^{\mathrm{light}}$.

2.1. **Solid abelian groups.** The heuristics behind the definition of solid abelian groups is the following simple idea: in non-archimedean analysis the summable sequences (i.e. sequences (a_n) for which $\sum_n a_n$ convergences) are precisely the null sequences (i.e. sequences (a_n) such that $a_n \to 0$). We have a condensed abelian group that parametrizes null sequences: let $\mathbb{N} \cup \{\infty\}$ be the profinite set obtained by adding a point in the positive direction of ∞, equivalently define $\mathbb{N} \cup \{\infty\} = \varprojlim_n \{1, \cdots, n\} \cup \{\infty\}$ where the transition map $\{1, \cdots, n+1\} \cup \{\infty\} \to \{1, \cdots, n\} \cup \{\infty\}$ sends $n+1$ to ∞.

Exercise 2.1.1. Let X be a topological space, show that continuous maps $\mathbb{N} \cup \{\infty\} \to X$ are the same as the datum of a sequence (a_n) together with a converging point $a_n \to a_\infty$.

Consider the condensed abelian group $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$. By construction it is the condensed abelian group that parametrizes *null sequences* in condensed abelian groups. Note that actually P is an algebra and that the inclusion $\mathbb{N} \to \mathbb{N} \cup \{\infty\}$ produces an algebra homomorphism

$$\mathbb{Z}[q] \to P.$$

We will denote $P = \mathbb{Z}[\widehat{q}]$ when P is considered as an algebra. Multiplication by q on $\mathbb{Z}[\widehat{q}]$ induces an endomorphism on P which we call "Shift". It is also induced from the map of profinite sets

$$\mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\} : \quad [n] \mapsto [n+1].$$

Exercise 2.1.2. In general being a null-sequence is not a property for a sequence but additional datum. Concretely this means the following: let A be a condensed abelian group, then the natural map

$$\phi : \mathrm{Hom}_{\mathbb{Z}}(P, A) \to A(\mathbb{N})$$

needs not to be injective.

(1) Show that, if A is quasi-separated, then ϕ is injective. Thus, for quasi-separated abelian groups being a null sequence is actually a property.
(2) Give an example of a condensed abelian group for which ϕ is not injective. Hint: consider the quotient $\underline{\mathbb{R}}/\mathbb{R}^\delta$ where $\mathbb{R}^\delta$ is endowed with the discrete topology.

The idea that a *non archimedean completion condition* is that null sequences become the same as invertible is formalized in the following definition:

Definition 2.1.3 ([RC24b, Definition 3.2.1])**.** A condensed abelian group $M \in \mathrm{Cond}(\mathsf{Ab})$ is said *solid* if the natural map

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M) \xrightarrow{1-\mathrm{Shift}^*} \underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M) \tag{2.1}$$

is an isomorphism. Equivalently, if the object

$$R\,\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[\widehat{q}]/(1-q), M) \cong 0$$

vanishes. We let $\mathsf{Solid} \subset \mathrm{Cond}(\mathsf{Ab})$ be the full subcategory of solid abelian groups.

Let us explain what the condition (2.1) encodes. The map $1 - \mathrm{Shift}^*$ sends a null sequence $(a_n)_n$ to the sequence $(a_n - a_{n+1})_n$. Therefore, if the map (2.1) has an inverse, it would be given by mapping a sequence (b_n) to the infinite sums $(\sum_{k \geq n} b_k)_n$ (this follows from a telescopic sum!).

The following theorem summarizes the main properties of solid abelian groups.

Theorem 2.1.4 ([RC24b, Theorem 3.2.3])**.** *Let* $\mathsf{Solid} \subset \mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}}$ *be the full subcategory of (light) solid abelian groups. The following hold:*

(1) Solid *is stable under limits, colimits and extensions in* $\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}}$*. In particular* Solid *it is an abelian category, and the inclusion has a left adjoint*

$$(-)^{\square} : \mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}} \to \mathsf{Solid}$$

called the solidification functor.

(2) *For all condensed abelian group M, and any $N \in \mathsf{Solid}$, the higher Ext objects $\underline{\mathrm{Ext}}^i(M,N)$ are solid abelian groups.*
(3) *$\mathbb{Z}$ is a solid abelian group. In particular any discrete group is a solid abelian group.*
(4) *The category Solid has a unique symmetric monoidal structure making $(-)^{\square}$ symmetric monoidal. Concretely, we have for $N, M \in \mathsf{Solid}$*

$$N \otimes^{\square} M = (N \otimes M)^{\square}.$$

(5) *$\mathbb{R}^{\square} = 0$. In particular, for any $\mathbb{R}$-module M one has $M^{\square} = 0$.*

The previous theorem has some fundamental content that we will explain now:

i. Part (1) of Theorem 2.1.4 is self explanatory: the category Solid is a thick Serre subcategory of condensed abelian groups stable under limits and colimits.
ii. Part (2) of the theorem says that solid abelian groups are enrich in condensed abelian groups (and in themselves).
iii. Part (3) says shows that Solid is non zero as it contains all the discrete modules. Actually we will see later that Solid is the smallest subcategory of condensed abelian groups containing $\mathbb{Z}$ and stable under limits and colimits. The existence of the left adjoint follows from a very general categorical phenomena called the reflection principle [RS22] and the adjoint functor theorem [Lur09, Corollary 5.5.2.9]; it says that a subcategory of a presentable category stable under limits and filtered enough colimits is itself presentable.
iv. Part (4) says that solidification only sees non-archimedean behaviour: it kills the real numbers and so any module over them. This is expected since there are null sequences on the real numbers which are not summable such as $(1/n)_{n\geq 1}$.

Exercise 2.1.5. In this exercise we prove parts (1)-(4) of Theorem 2.1.4.

(1) Use the fact that $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ is internally projective to formally deduce parts (1) and (2) of Theorem 2.1.4.
(2) Let A be a discrete abelian group. Compute $\underline{\mathrm{Hom}}(P, A)$ and show that the map $1 - \mathrm{Shift}^*$ is an isomorphism proving part (3).
(3) Let Q be a condensed abelian group such that $Q^{\square} = 0$, prove that for any condensed abelian group M one has $(Q \otimes M)^{\square} = 0$. Deduce that the subcategory $\mathscr{K}$ of condensed abelian groups killed by the solidification functor is a tensor ideal and so that $\otimes$ localizes to a symmetric monoidal structure $\otimes_{\square}$ in Solid.

Exercise 2.1.6. In this exercise we prove part (5) of Theorem 2.1.4.

i. Show that $\mathbb{R}^{\square}$ is an algebra. Thus, to prove that it vanishes it suffices to show that the unit $\mathbb{Z} \to \mathbb{R}^{\square}$ is zero (same for any unit in $\mathbb{R}$).
ii. Consider the null sequence $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots)$ in $\mathbb{R}$ defining a map $f : P \to \mathbb{R}$. Show that there is a unique map $g : P \to \mathbb{R}^{\square}$ making the following diagram commute

$$\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle 1-\mathrm{Shift}^*}\downarrow & & \downarrow \\ P & \xrightarrow{g} & \mathbb{R}^{\square}. \end{array}$$

iii. Let $[0] : \mathbb{Z} \to P$ be the inclusion in the zeroth component, and consider the element $x \in \mathbb{R}^{\square}$ given by $g \circ [0](1)$. Prove that $x = 2 + x$ and so that $2 = 0$. Deduce that $\mathbb{R}^{\square} = 0$. Hint: consider the maps

$$F : \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] : \quad [n] \mapsto [2n+1] + [2n+2]$$
$$G : \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] : \quad [n] \mapsto [2n+1]$$

and show that there is a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & P \\ {\scriptstyle 1-\mathrm{Shift}^*}\downarrow & & \downarrow{\scriptstyle 1-\mathrm{Shift}^*} \\ P & \xrightarrow{G} & P. \end{array}$$

Let us continue the description of the category of solid abelian groups by passing to derived categories. Instead to directly working with the derived category of solid abelian groups we will work with a variant:

Definition 2.1.7. Let $D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})^{\square} \subset D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})^{\square}$ be the full subcategory consisting on objects C such that

$$R\,\underline{\mathrm{Hom}}(P,C) \xrightarrow{1-\mathrm{Shift}^*} R\,\underline{\mathrm{Hom}}(P,C)$$

is an isomorphism.

Proposition 2.1.8. *The following hold:*

(1) $D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})^{\square}$ *is stable under all limits and colimits in* $D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})$ *(eq. the underlying homotopy category is stable under cones, direct sums and products).*
(2) $D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})^{\square}$ *consists on the full subcategory of complexes* C *such that* $H^i(C) \in \mathsf{Solid}$ *for all* $i \in \mathbb{Z}$.
(3) *The fully faithful inclusion has a left adjoint*
$$(-)^{L\square} : D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}}) \to D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})^{\square}.$$
(4) *The category* $D(\mathrm{Cond}(\mathsf{Ab})^{\mathrm{light}})^{\square}$ *has a unique symmetric monoidal structure, denoted by* $\otimes^L_{\square}$, *making* $(-)^{L\square}$ *a symmetric monoidal functor.*

Exercise 2.1.9. Prove Proposition 2.1.8 by following the same arguments of Exercise 2.1.6. Hint: use [NS18, Proposition A.5] to obtain the symmetric monoidal structure.

Remark 2.1.10. The reader might ask why we introduced the category $D(\mathrm{Cond}(\mathsf{Ab}))^{\square}$ instead of directly working with $D(\mathsf{Solid})$. The reason is that there are two natural candidates for a derived category of solid complexes: 1) the derived category of solid abelian groups, or 2) the condensed abelian complexes with solid abelian groups. In the next propositions we will see that both actually agree.

In order to state more properties of the category of solid abelian groups we first need to compute the *solid measures* attached to profinite sets. Motivated from the notation of analytic rings (to be discussed later) we make the following definition:

Definition 2.1.11. Let S be a profinite set, we denote

$$\mathbb{Z}_{\square}[S] := \mathbb{Z}[S]^{L\square}.$$

Theorem 2.1.12 ([RC24b, Theorem 3.3.1])**.** *Let* $S = \varprojlim_n S_n$ *be a light profinite set written as a countable limit of finite sets. The natural map*

$$\mathbb{Z}[S] \to \varprojlim_n \mathbb{Z}[S_n]$$

induces an equivalence

$$\mathbb{Z}_{\square}[S] \xrightarrow{\sim} \varprojlim_n \mathbb{Z}[S_n].$$

Remark 2.1.13. Let $S = \varprojlim S_n$ be a light profinite set written as a countable limit of finite sets. One has that

$$\varprojlim_n \mathbb{Z}[S_n] = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$$

is the space of $\mathbb{Z}$-valued measures on $\mathbb{Z}$. This point of view using measures was the initial starting point in defining the category of (general) solid abelian groups in [CS19]. The techniques used in *loc. cit.* are more sofisticated and require a rather involved devisage to Ext groups of locally compact abelian groups seen as condensed abelian groups. The advantage of the light approach is that solid abelian groups are now defined using a *property* and what is left to do is to *compute* the measures $\mathbb{Z}_{\square}[S]$.

Exercise 2.1.14. Let S be a light profinite set. Show that $C(S,\mathbb{Z})$ is a free abelian group. Deduce that $\underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$ is isomorphic to a product of copies of $\mathbb{Z}$. Hint: find a countable dense subspace of S.

Exercise 2.1.15. In this exercise we shall prove Theorem 2.1.12.

i. Show that there is an injective map $P\to\mathbb{Z}[S]$ such that $P^{L\square}\to\mathbb{Z}[S]^{L\square}$ is an equivalence. Hint: follow the proof of [RC24b, Lemma 3.3.2].
ii. Let $\prod_{\mathbb{Z}}^{\mathrm{bnd}}\mathbb{Z}=\bigcup_n\prod_{\mathbb{N}}(\mathbb{Z}\cap[-n,n])\subset\prod_{\mathbb{N}}\mathbb{Z}$ be the condensed set of bounded sequences of integers. Prove that the sequence $(e_n)_{n\in\mathbb{N}}$ with $e_n=(0,\ldots,1,0,\ldots)$ with 1 in the n-th component is a null sequence. Deduce a map $P\to\prod_{\mathbb{Z}}^{\mathrm{bnd}}\mathbb{Z}$.
iii. Show that the natural map
$$P^{L\square}\to(\prod_{\mathbb{Z}}^{\mathrm{bnd}}\mathbb{Z})^{L\square}$$
is an equivalence. Hint: see part (i.) or [RC24b, Lemma 3.3.3]
iv. Show that the quotient $\prod_{\mathbb{N}}\mathbb{Z}/\prod_{\mathbb{N}}^{bnd}\mathbb{Z}$ has a structure of real vector space. Deduce that the natural map
$$\prod_{\mathbb{N}}^{\mathrm{bnd}}\mathbb{Z}\xrightarrow{\sim}\prod_{\mathbb{N}}\mathbb{Z}$$
is an isomorphism. Hint: show that it can be written as $\prod_{\mathbb{N}}\mathbb{R}/\prod_{\mathbb{N}}^{\mathrm{bnd}}\mathbb{R}$ for a suitable definition of $\prod_{\mathbb{N}}^{\mathrm{bnd}}\mathbb{R}$.
v. Show Theorem 2.1.12. Hint: keep track of the map constructed in part (i).

We finally can related the category of Definition 2.1.7 with the derived category of solid abelian groups.

Theorem 2.1.16 ([RC24b, Theorem 3.2.3])**.** *Let* Solid *be the category of solid abelian groups and* $D(\mathrm{Cond}(\mathsf{Ab}))^{\square}\subset D(\mathrm{Cond}(\mathsf{Ab}))$ *the full subcategory of complexes* C *with solid cohomology groups. The following hold:*

(1) *The natural map* $D(\mathsf{Solid})\to D(\mathrm{Cond}(\mathsf{Ab}))$ *factors through* $D(\mathrm{Cond}(\mathsf{Ab}))^{\square}$ *and induces an equivalence of* ∞*-categories*
$$D(\mathsf{Solid})\xrightarrow{\sim}D(\mathrm{Cond}(\mathsf{Ab}))^{\square}$$
(2) *The functor* $(-)^{L\square}$ *is the left derived functor of* $(-)^{\square}$*.*
(3) *Given countable sets* I *and* J *the natural map*
$$\prod_I\mathbb{Z}\otimes_{\square}^L\prod_J\mathbb{Z}\xrightarrow{\sim}\prod_{I\times J}\mathbb{Z}$$
is an isomorphism.
(4) *The tensor product* $\otimes_{\square}^L$ *is the left derived functor of* $\otimes_{\square}$*.*
(5) *The object* $P^{L\square}\cong\prod_{\mathbb{N}}\mathbb{Z}$ *is a compact projective generator of* Solid*.*

Exercise 2.1.17. In this exercise we will prove Theorem 2.1.16.

i. Show that the natural functor $D(\mathsf{Solid})\to D(\mathrm{Cond}(\mathsf{Ab}))$ factors through complexes with solid cohomology.
ii. Show that $\prod_{\mathbb{N}}\mathbb{Z}$ is a compact projective generator of Solid proving (5).

iii. Show that for solid modules N, M the natural map
$$R\mathrm{Hom}_{\mathsf{Solid}}(N, M) \to R\mathrm{Hom}_{\mathbb{Z}}(N, M)$$
is an equivalence. Deduce that the functor $D(\mathsf{Solid}) \to D(\mathrm{Cond}(\mathsf{Ab}))$ is fully faithful. Hint: use a projective resolution in Solid.
iv. Show that if $C \in D(\mathrm{Cond}(\mathsf{Ab}))$ has solid cohomology then it lies in the essential image of $D(\mathsf{Solid})$. Deduce part (1).
v. Prove that $(-)^{L\square}$ is the left derived functor of $(-)^{\square}$. Hint: to show that a functor of derived categories is a derived functor it suffices to find a generating family of acyclic objects. Deduce part (2).
vi. Prove part (3). Hint: what is $\mathbb{Z}_{\square}[S] \otimes^L_{\square} \mathbb{Z}_{\square}[T]$ for S and T light profinite sets?
vii. Prove part (4). Hint: same as in (iii).

Remark 2.1.18. so far Theorems 2.1.4, 2.1.12 and 2.1.16 hold for general solid abelian groups, though the profs are different and involve a long devisage to Ext groups of locally compact abelian groups, see [CS19].

The next proposition only holds in the light set up:

Proposition 2.1.19. *The solid abelian group $\prod_{\mathbb{N}} \mathbb{Z}$ is flat.*

Exercise 2.1.20. Read the proof of Proposition 2.1.19 in [RC24b, Section 3.4].

Exercise 2.1.21. Let Q be a pseudo-compact object in $D_{\geq 0}(\mathsf{Solid})$, i.e. an object Q such that $R\,\underline{\mathrm{Hom}}(Q, -)$ commutes with filtered colimits in $D_{\geq 0}(\mathsf{Solid})$. Prove that Q can be represented by a connective complex whose terms are isomorphic to $\mathbb{Z}_{\square}[S]$ for S a profinite set.

Exercise 2.1.22. Prove the following solid identities:

- $\mathbb{Z}[[X]] \otimes^L_{\square} \mathbb{Z}[[Y]] = \mathbb{Z}[[X, Y]]$
- $\mathbb{Z}[[X]]/(X - p) = \mathbb{Z}_p$.
- Let I be a countable set. Then $\mathbb{Z}_p \otimes^L_{\square} \prod_I \mathbb{Z} = \prod_I \mathbb{Z}_p$.
- More generally, let I be a countable set and let Q be a pseudo-compact object in $D_{\geq 0}(\mathsf{Solid})$. Prove that
$$Q \otimes^L_{\square} \prod_I \mathbb{Z} = \prod_I Q.$$
- $\mathbb{Z}_p \otimes^L_{\square} \mathbb{Z}_{\ell} = 0$ if $p \neq \ell$ and $\mathbb{Z}_p \otimes^L_{\square} \mathbb{Z}_p = \mathbb{Z}_p$.
- Let I be a index set and let $\widehat{\bigoplus}_I \mathbb{Z}_p$ be the p-completion of $\bigoplus_I \mathbb{Z}_p$. Then
$$\widehat{\bigoplus}_I \mathbb{Z}_p \otimes^L_{\square} \widehat{\bigoplus}_J \mathbb{Z}_p = \widehat{\bigoplus}_{I \times J} \mathbb{Z}_p.$$
Hint: write a presentation of the completed direct sum in terms of products of $\mathbb{Z}_p$.
- (Harder) Let N, M be connective derived p-adically complete objects in $D_{\geq 0}(\mathsf{Solid})$. Prove that $N \otimes^L_{\square} M$ is derived p-complete.

2.2. **Solid quasi-coherent sheaves.** So far we have studied the category of solid abelian groups. A natural question is whether for a commutative discrete ring A there is an analogue category of *solid A-modules*. The answer is affirmative and we will discuss a sketch of its construction.

For a condensed ring R we shall write $D(R)$ for its derived category of condensed R-modules. Since $R\,\mathrm{Hom}_R(R, R) = R(*)$, we have a fully faithful embedding $D(R(\star)) \to D(R)$ from the derived category of $R(*)$-modules into condensed R-modules. We denote the essential image by $D(R)^{\delta}$ and call it the derived category of discrete R-modules.

Remark 2.2.1. Given a condensed ring R and a discrete R-module M it is not true that M is discrete as a condensed set. However, it is discrete *relative to R* in the sense that the only non-trivial condensed structure on M arises from that of R. If R is a discrete ring then a discrete R-module is also discrete as condensed set.

We will make a change of notation for the category of solid abelian groups.

Definition 2.2.2. We will denote by $D(\mathbb{Z}_\square)$ for the derived category of solid abelian groups $D(\mathsf{Solid})$. More generally, let R be a solid ring, i.e. a commutative algebra object in Solid. We denote by $D((R,\mathbb{Z})_\square)$ the derived category of condensed R-modules whose underlying condensed abelian group is solid. We also say that the objects in $D((A,\mathbb{Z})_\square)$ are $(A,\mathbb{Z})_\square$-modules. Note that the category $D((A,\mathbb{Z})_\square)$ has a natural symmetric monoidal structure which we denote by $\otimes^L_{(A,\mathbb{Z})_\square}$.

Definition 2.2.2 provides a notion of solid A modules for discrete algebras (even for any solid algebra!). However, these solid modules are only complete over $\mathbb{Z}$ and not over A in general. Indeed, the free $(A,\mathbb{Z})_\square$-module generated by the light profinite set S is

$$(A,\mathbb{Z})_\square[S] = A \otimes_{\mathbb{Z}} \mathbb{Z}_\square[S].$$

For instance, if $A=\mathbb{Z}[T]$ is a polynomial algebra we have that

$$(\mathbb{Z}[T],\mathbb{Z})_\square[S] \cong (\prod_I \mathbb{Z})[T]$$

for some countable index set I. Instead we would like to have a completed version of $\mathbb{Z}[T]$ such that

$$\mathbb{Z}[T]_\square[S] \cong \prod_I \mathbb{Z}[T].$$

This is actually possible, and the key notion is that of T-*summable sequences*:

Definition 2.2.3. Let A be a discrete ring and let $a\in A$. Let $P=\mathbb{Z}[\mathbb{N}\cup\{\infty\}]/(\infty)$. An object $M\in D((A,\mathbb{Z})_\square)$ is a-summable if the natural map

$$R\,\underline{\mathrm{Hom}}_A(A\otimes^L_\square P^\square, M) \xrightarrow{1-a\mathrm{Shift}^*} R\,\underline{\mathrm{Hom}}_A(A\otimes^L_\square P^\square, M)$$

is an isomorphism. We say that M is A-solid if it is a-summable for all $a\in A$. We let $D(A_\square)\subset D((A,\mathbb{Z})_\square)$ be the full subcategory of A-solid elements.

Remark 2.2.4. We preferred to give the derived definition of A-solid modules immediately since from the perspective of analytic geometry it is much more natural to work directly with ∞-derived categories instead of abelian categories. Nevertheless, the analogues of Theorems 2.1.4 and 2.1.16 will hold for solid A-modules so the reader that feels more comfortable with abelian categories can consider the abelian version of Definition 2.2.3 instead.

Remark 2.2.5. The idea behind Definition 2.2.3 is the following. A null sequence $(x_n)_{n\in\ltimes}$ will be a-summable precisely when $\sum_n x_n a^n$ converges. In the case of $a=1$ we recover the definition of solid abelian groups, but for more general elements in $a\in A$ we get a class of modules which are A-*complete* in a suitable sense.

The analogue of Theorems 2.1.4 and 2.1.16 hold for solid A-modules:

Theorem 2.2.6. *Let A be a discrete ring, the following holds:*

1. *The full subcategory $D(A_\square)\subset D((A,\mathbb{Z})_\square)\subset D(\mathrm{Cond}(\mathsf{Mod}(A)))$ of solid A-modules is stable under limits and colimits. In particular it is presentable and the inclusion admits a left adjoint which we write as*
$$A_\square\otimes^L_{\underline{A}} - : D(\mathrm{Cond}(\mathsf{Mod}(A)))\to D(A_\square)$$
called the A-solidification functor.
2. *For any condensed A-module M and any objects $N\in D(A_\square)$ one has $R\,\underline{\mathrm{Hom}}_A(M,N)\in D(A_\square)$.*
3. *The category $D(A_\square)$ has a unique symmetric monoidal structure denoted as $\otimes^L_{A_\square}$ making $A_\square\otimes_{\underline{A}}-$ symmetric monoidal.*

(4) *The functor $A_{\square}\otimes^L_{\underline{A}}-$ sends connective objects to connective objects. In particular, $D(A_{\square})$ has a well defined t-structure. Furthermore $D(A_{\square})$ is the derived category of its heart*
(5) $\otimes^L_{A_{\square}}$ *and* $A_{\square}\otimes^L_{\underline{A}}-$ *are the left derived functors of their heart.*
(6) *A is A-solid.*
(7) *Let $S=\varprojlim_n S_n$ be a light profinite set written as a limit of finite sets S_n, then*

$$A_{\square}[S] := A\otimes^L_{\underline{A}}(A[S]) = \varinjlim_{B\subset A}\varprojlim_n B[S_n]$$

where B runs over all subrings of A which are finite type $\mathbb{Z}$-algebras.

Remark 2.2.7. We will not expend too much time in proving Theorem 2.2.6 as its proof is essentially the same as for solid modules. The reason why both proves are so closely related is because they fit into the more general definition of an *analytic ring*. In other words, most of Theorem 2.2.6 is saying that $A_{\square}$ is an analytic ring structure on A. We will briefly review the theory of analytic geometry and analytic stacks in the next lecture.

Remark 2.2.8. Let A be a discrete ring. One might wonder if there is a suitable definition of solid A-module so that for $S=\varprojlim_n S_n$ light profinite one has

$$A_{\square}[S] = \varprojlim_n A[A_n]. \tag{2.2}$$

Theorem 2.2.6 says that this holds for the given definition of $A_{\square}$ when A is a $\mathbb{Z}$-algebra of finite type. It turns out that one can define suitable analytic ring structure on rings *essentially of finite type* satisfying (2.2), see [BK24], we call those analytic rings *ultrasolid structures on A*. For general discrete rings there are no analytic rings satisfying (2.2), however one can construct a category of ultrasolid A-modules *by hand* that will satisfy plenty of good properties but which is not a full subcategory of condensed A-modules, see [MA24].

Exercise 2.2.9. In order to understand the category of solid A-modules it suffices to understand the *universal example*: $A=\mathbb{Z}[T]$ and T-summable modules (any other case will be a colimit of this in the world of analytic rings). To make this category more precise let us recall the definition of being T-summable: let $P=\mathbb{Z}[\mathbb{N}\cup\{\infty\}]/(\infty)$. A $(\mathbb{Z}[T],\mathbb{Z})_{\square}$-module M is T-summable is the natural map

$$\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T]\otimes_{\square}P^{\square},M)\xrightarrow{1-T\mathrm{Shift}^*}\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T]\otimes_{\square}P^{\square},M) \tag{2.3}$$

is an equivalence. Recall that $\mathbb{Z}[\widehat{q}]=P$ has an algebra structure, by Theorem 2.1.12 one has $\mathbb{Z}[\widehat{q}]^{\square}=\mathbb{Z}[[q]]$. Note that

$$(\mathbb{Z}[T]\otimes_{\square}\mathbb{Z}[[q]])/(1-Tq)=\mathbb{Z}((T^{-1})) \tag{2.4}$$

is a Laurent power series ring in the variable T^{-1}. Then, we can rewrite (2.3) as

$$R\,\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),M)=0. \tag{2.5}$$

(1) Prove that $\mathbb{Z}((T^{-1}))$ is a compact $(\mathbb{Z}[T],\mathbb{Z})$-module.
(2) Show that $\mathbb{Z}((T^{-1}))$ is an idempotent $(\mathbb{Z}[T],\mathbb{Z})_{\square}$-algebra.
(3) Show that

$$R\,\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),\mathbb{Z}[T])=0.$$

(4) Let I be a countable set. Show that the natural map

$$\prod_I(\mathbb{Z}[T])/((\prod_I\mathbb{Z})\otimes_{\mathbb{Z}_{\square}}\mathbb{Z}[T])\xrightarrow{\sim}(\prod_I\mathbb{Z}((T^{-1})))/((\prod_I\mathbb{Z})\otimes_{\mathbb{Z}_{\square}}\mathbb{Z}((T^{-1})))$$

is an isomorphism of $(\mathbb{Z}[T],\mathbb{Z})$-modules.
(5) Give a proof of Theorem 2.2.6 for $A=\mathbb{Z}[T]$.
(6) Extend the proof of Theorem 2.2.6 to polynomial algebras for finitely variables.

Hint: Eq. (2.4).

Exercise 2.2.10. Let A be a $\mathbb{Z}$-algebra of finite type and let $\mathbb{Z}[T_1,\dots,T_n]\to A$ be a surjective map. Show that

$$\prod_{\mathbb{N}} \mathbb{Z}[T_1,\dots,T_n]\otimes^L_{\mathbb{Z}[T_1,\dots,T_n]} A = \prod_{\mathbb{N}} A.$$

Deduce that an A-module is A-solid if and only if it is T_i-summable for $i=1,\dots,n$. Then show that solid A-modules are the same as A-modules in $D(\mathbb{Z}[T_1,\dots,T_n]_{\square})$. Finally deduce Eq. (2.4) for A.

Exercise 2.2.11. Let $A=\varinjlim_i A_i$ be a discrete ring written as a colimit of finite type subalgebras. Show that

$$A_{\square}[S]=\varinjlim_i A_{i,\square}[S].$$

Then prove Eq. (2.4) for A.

We finish this section with the construction of the base change of solid modules over rings (this construction will be better explained in the context of analytic rings):

Theorem 2.2.12. *Let $A\to B$ be a morphism of discrete rings, then there is a unique colimit preserving functor*

$$B_{\square}\otimes^L_{A_{\square}} - : D(A_{\square})\to D(B_{\square})$$

so that the following diagram commutes

$$\begin{array}{ccc} D(\underline{A}) & \xrightarrow{B\otimes^L_{\underline{A}}-} & D(\underline{B}) \\ {\scriptstyle A_{\square}\otimes^L_{\underline{A}}-}\downarrow & & \downarrow{\scriptstyle B_{\square}\otimes^L_{\underline{B}}-} \\ D(A_{\square}) & \xrightarrow{B_{\square}\otimes^L_{A_{\square}}-} & D(B_{\square}). \end{array}$$

Furthermore, the functor $B_{\square}\otimes^L_{A_{\square}}-$ is symmetric monoidal and the left derived functor of its heart. For a light profinite set S we have

$$B_{\square}\otimes^L_{A_{\square}} A_{\square}[S]=B_{\square}[S].$$

2.3. **Serre duality.** Let X be a smooth proper scheme over a field K of dimension d and let $f:X\to \operatorname{Spec} K$ be the structural map. The classical statement of Serre duality says the following: let V be a vector bundle on X, then there is a natural isomorphism of cohomology groups

$$H^i(X,V)^{\vee}\cong H^{d-i}(X,V^{\vee}\otimes\Omega^d_{X/K}).$$

It has been generalized by Grothendieck to proper smooth maps $f:X\to S$ of schemes of relative dimension d and quasi-coherent sheaves. The best way to state the theorem is using the theory of 6-*functors*. Let $f^*:D^{qc}(S)\to D^{qc}(X)$ be the pullback functor of derived quasi-coherent sheaves, it has a right adjoint given by the pushforward functor f_*. Then Grothendieck-Serre duality states that f_* has a further right adjoint $f^!$, and that $f^!$ and f^* are related via the formula

$$f^!=\Omega^d_{X/S}[d]\otimes_{\mathscr{O}_X} f^*.$$

The classical proof of Grothendieck-Serre duality is quite involved and has had some sign mistakes in the literature that have been carefully addressed, cf. [Har66, Con00]. In the case of smooth projective varieties it can be deduced from a long dévisage from the case of the projective space where cycle and trace maps are explicitly constructed.

Thanks to the theory of solid modules over discrete rings, and the abstract theory of six functor formalisms [Man22b, Sch23, HM24] one can now finally provide a conceptual proof of Grothendieck-Serre duality requiring very few concrete computations (the most difficult part being developing the necessary technology). What is more fantastic are the following facts:

i. One can actually built up a full six functor formalism of *solid quasi-coherent sheaves on schemes* locally of finite type over $\mathbb{Z}$ so that given a map $f : X \to S$ there is a natural "cohomology with compact supports" $f_!$. One then defines the exceptional pullback $f^!$ to be the right adjoint of $f_!$.
ii. Given $f : X \to S$ *any* smooth map of dimension d one has $f^! = \Omega^d_{X/S}[d] \otimes f^*$.
iii. The arguments in the proof of Grothendieck-Serre duality for schemes also apply for rigid spaces and complex manifolds. In other words, there is a *universal proof* of Grothendieck Serre duality in different analytic geometries.

Remark 2.3.1. Classically there is no such a general statement for Serre duality for smooth maps. Actually, there cannot be such an statement even for Zariski open immersions. Indeed, let A be a $\mathbb{Z}$-algebra of finite type and consider $A[\frac{1}{f}]$. Then, classically one has that

$$(\prod_I A) \otimes_A A[\frac{1}{f}] \neq \prod_I A[\frac{1}{f}].$$

Thus, the base change $A[\frac{1}{f}] \otimes_A -$ cannot be a right adjoint (as it does not preserve limits) and so there is no a reasonable Serre duality. However, if we now look at solid modules the problem gets solve! Indeed, we now have by design

$$(\prod_I A) \otimes_{A_\square} A[\frac{1}{f}]_\square = \prod_I A[\frac{1}{f}].$$

In other words, if we want open immersions to satisfy Serre duality we are forced to take complete tensor products, which then implies that our modules must be endowed with a suitable notion of topology which in our context means that they must be solid modules.

The proof of the Grothencieck-Serre duality can be separated in three parts:

i. Constructing the 6-functor formalism of solid quasi-coherent sheaves.
ii. Proving that a smooth map $f : X \to S$ is cohomologically smooth (i.e. that $f^! = f^!\mathscr{O}_S \otimes f^*$); thanks to the abstract 6-functor theory this reduces to open immersions and the affine line $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$.
iii. The identification of the dualizing sheaf $f^!\mathscr{O}_S = \Omega^d_{X/S}[d]$ which is done by a deformation to the normal cone argument, see [CS22, Lecture XIII].

Explaining the whole proof of Serre duality is too ambitious for these notes. In the rest of the section we define the ∞-category of solid quasi-coherent sheaves on a scheme X and give some exercises which are key in the proof of part (ii). First, we state Zariski descent:

Theorem 2.3.2. *Let A be a discrete ring, $X = \operatorname{Spec} A$ and let $\{U_i \subset X\}_{i=1}^n$ be a Zariski open cover. Let $\mathscr{J}$ be the poset of finite intersections of $\{U_i\}_{i=1}^n$. Then the natural functor of ∞-categories*

$$D(A_\square) \to \varprojlim_{J \in \mathscr{J}} D(\mathscr{O}(U_J)_\square)$$

is an equivalence, where $U_J = \cap_{j\in J} U_j$. In other words, solid quasi-coherent sheaves satisfy Zariski descent.

Remark 2.3.3. In Theorem 2.3.2 we used the language of ∞-categories in order to state Zariski descent of solid quasi-coherent sheaves. In classical algebraic geometry going that far is not necessary and one has descent for abelian categories; the reason being that base change along open immersions **is flat**. However, **flatness** along Zariski open immersions **fails** for solid modules; to give an example consider $\mathbb{Z}[T] \to \mathbb{Z}[T^{\pm 1}]$, then

$$\mathbb{Z}[T^{\pm 1}]_\square \otimes^L_{\mathbb{Z}[T]_\square} \mathbb{Z}[[T]]/\mathbb{Z}[T] = \mathbb{Z}[T^{\pm 1}][1].$$

This makes impossible to obtain good descend properties only using abelian categories.

Definition 2.3.4. Let X be a scheme, its ∞-derived category of solid quasi-coherent sheaves $D(X_{\square})$ is the sheaf on X for the Zariski topology mapping an open affine subspace $U \subset X$ to the ∞-category

$$U \mapsto D(\mathscr{O}(U)_{\square}).$$

Remark 2.3.5. By Remark 2.3.3 there is no a good notion of abelian category of solid quasi-coherent sheaves on a scheme X. However, if X is of finite type over an affine scheme, one can still define a subcategory of bounded objects.

We can informally state Serre duality in the following way:

Theorem 2.3.6 (Clausen-Scholze)**.** *There is a unique six functor formalism for solid quasi-coherent sheaves on schemes so that:*

i. *For open immersions of affine schemes $f : U \to X$ the functor $f_!$ is the left adjoint of f^*.*
ii. *For a finite morphism $f : Z \to X$ of affine schemes the functor $f_! = f_*$ is the right adjoint of f^*.*

Furthermore the following hold for a morphism of schemes $f : X \to S$:

(1) *If f is étale then $f^! = f^*$.*
(2) *If f is proper then $f_* = f_!$.*
(3) *If f is smooth of relative dimension d then $f^! = f^!\mathscr{O}_S \otimes f^*$. Furthermore, there is a natural equivalence $f^!\mathscr{O}_S = \Omega^d_{X/S}[d]$.*

Exercise 2.3.7. In this exercise we will introduce the theory of locales and smashing spectrum of presentably symmetric monoidal stable ∞-categories. This language is extremely useful when working in analytic geometry, we refer to [CS22] or [RC24b] for more details.

(1) Let $\mathscr{C}$ be a symmetric monoidal presentably stable ∞-category with unit 1 and tensor $\otimes$. An idempotent algebra in $\mathscr{C}$ is the datum of a map $\mu : 1 \to A$ in $\mathscr{C}$ such that the natural map

$$A \xrightarrow{\mu \otimes \mathrm{id}_A} A \otimes A$$

is an equivalence. It turns out that an idempotent algebra as previously defined has a natural structure of commutative algebra in $\mathscr{C}$. Show that given two idempotent algebras A and B the mapping space of morphisms from A to B preserving the unit is either contractible or empty. Show that the space is contractible if and only if $B \xrightarrow{\sim} B \otimes A$ is an equivalence. Deduce that the ∞-category of idempotent algebras in $\mathscr{C}$ is just a poset.

(2) We let $\mathscr{S}(\mathscr{C})$ be the *opposite* of the poset of idempotent algebras in $\mathscr{C}$, we call $\mathscr{S}(\mathscr{C})$ the *smashing spectrum of* $\mathscr{C}$. Prove that $\mathscr{S}(\mathscr{C})$ is a locale. More precisely, by identifying closed subspaces of $\mathscr{S}(\mathscr{C})$ with idempotent algebras of $\mathscr{C}$, prove that it is a locale with the following operations:

i. The empty subspace is associated to the idempotent algebra 0. The total space is associated to the idempotent algebra 1.
ii. Given A and B idempotent algebras, their *intersection* correspond to the algebra $A \cap B = A \otimes B$. More generally, given a diagram $\{A_i\}_{i\in I}$ of idempotent algebras, their intersection correspond to the colimit $\bigcap_i A_i = \varinjlim_i A_i$.
iii. Given A and B idempotent algebras, their *union* correspond to the idempotent algebra

$$A \cup B = A \bigsqcup_{A\cap B} B = A \times_{A\otimes B} B.$$

More generally, given a finite family of idempotent algebras $\{A_i\}_{i\in I}$ one defines

$$\bigcup_i A_i = \varprojlim_{J \in \mathscr{J}} A_J$$

where $\mathscr{J}$ runs over finite subsets of I and for $J \in \mathscr{J}$ one has $A_J = \bigotimes_{j\in J} A_j$.

(3) Let A be an idempotent algebra in $\mathscr{C}$ wich closed subspace $Z \in \mathscr{S}(\mathscr{C})^{\mathrm{op}}$ and open complement $U \in \mathscr{S}(\mathscr{C})$ (i.e. A is the underlying object of Z and U). We can associate presentably stable ∞-categories to Z and U as follows:
   i. We define $\mathscr{C}(Z) = \mathrm{Mod}_A(\mathscr{C})$ to be the category of A-modules in $\mathscr{C}$. Show that $\mathscr{C}(Z)$ can be identified with the full subcategory of $\mathscr{C}$ of objects M such that the map $M \xrightarrow{\mu\otimes \mathrm{id}_M} A\otimes M$ is an equivalence (equivalently with those M such that $\underline{\mathrm{Hom}}_{\mathscr{C}}(A, M) \xrightarrow{\mu^*} M$ is an equivalence).
   ii. We define the ∞-category $\mathscr{C}(U)$ to be the quotient $\mathscr{C}(U) := \mathscr{C}/\mathscr{C}(Z)$. Prove that the localization functor $\mathscr{C} \to \mathscr{C}(U)$ has fully-faithful left and right adjoints. Hint: If we call $\iota_* : \mathscr{C}(Z) \to \mathscr{C}$ the forgetful functor and ι^* and $\iota^!$ are the left and right adjoints, and if $j^* : \mathscr{C} \to \mathscr{C}(U)$ is the localization functor with left and right adjoints $j_!$ and j_* respectively, show that there are excision sequences
$$j_! j^* \to \mathrm{id} \to \iota_*\iota^*$$
and
$$\iota_*\iota^! \to \mathrm{id} \to j_*j^*.$$
Give precise formulas for the previous excision sequences.

We call the localization maps $\iota^* : \mathscr{C} \to \mathscr{C}(Z)$ and $j^* : \mathscr{C} \to \mathscr{C}(U)$ *closed and open* localizations.

(4) Let $f^* : \mathscr{C} \to \mathscr{D}$ be a morphism of presentably symmetric monoidal stable ∞-categories. Show that f^* induces a *continuous morphisms of locales* $f^{-1} : \mathscr{S}(\mathscr{C}) \to \mathscr{S}(\mathscr{D})$.

(5) Show that the following are equivalent for a morphism $f^* : \mathscr{C} \to \mathscr{D}$ of presentably symmetric monoidal stable ∞-categories:
   i. f^* is isomorphic to an open localization $j^* : \mathscr{C} \to \mathscr{C}(U)$.
   ii. f has a fully-faithful left adjoint $f_!$ satisfying projection formula: for $N \in \mathscr{C}$ and $M \in \mathscr{D}$ the natural map
$$f_!(M \otimes f^*N) \to f_!M \otimes N$$
is an equivalence.

Deduce that an open localization of an open localization is an open localization.

(6) Show that the following are equivalent for a morphism $f^* : \mathscr{C} \to \mathscr{D}$ of presentably symmetric monoidal stable ∞-categories:
   i. f^* is isomorphic to a closed localization $\iota^* : \mathscr{C} \to \mathscr{C}(Z)$.
   ii. f^* has a fully-faithful colimit preserving right adjoint f_* satisfying projection formula: for $N \in \mathscr{C}$ and $M \in \mathscr{D}$ the natural map
$$f_*M \otimes N \to f_*(M \otimes f^*N)$$
is an equivalence.

Deduce that a closed localization of a closed localization is a closed localization.

(7) Let $\mathscr{C}$ be as before and let $\{U_i\}_{i\in I}$ be a family of open subspaces of $\mathscr{S}(\mathscr{D})$ with closed complements $\{Z_i\}_{i\in I}$. By definition $\{U_i\}_{i\in I}$ is a *cover* if $\bigcap_i Z_i = \emptyset$, that is, if $\varinjlim_i A_i = \emptyset$ where A_i is the idempotent algebra attached to Z_i. Let $\mathscr{J}$ be the poset of finite intersections of the U_i's. Prove that the natural map
$$F : \mathscr{C} \to \varprojlim_{J\in\mathscr{J}} \mathscr{C}(U_J)$$
is an equivalence, where $U_J = \cap_{j\in J} U_j$. In other words, that we have *descend* for the natural topology of the locale $\mathscr{S}(\mathscr{C})$. Hint: the functor F as a right adjoint G. Compute this right adjoint and show that the natural transformations
$$1 \to GF \text{ and } FG \to 1$$
are equivalences.

(8) Similarly, let $\{Z_i\}_{i\in I}$ be a finite collection of closed subspaces of $\mathscr{S}(\mathscr{C})$ whose union is the total space. Let $\mathscr{J}$ be the finite poset of subspaces of I. Show that the natural map

$$F : \mathscr{C} \to \varprojlim_{J\in\mathscr{J}} \mathscr{C}(Z_J)$$

is an equivalence. Hint: let A_i be the idempotent algebra of Z_i, prove that the algebra $\prod_{i=1}^n A_i$ is descendable in $\mathscr{C}$ (see [Mat16] for the definition of descendability).

Exercise 2.3.8. In this exercise we use the theory of smashing spectrum to prove Zariski descent for solid quasi-coherent sheaves. Let A be a discrete ring and $\{U_i \to \operatorname{Spec} A\}_i$ an open Zariski cover.

(1) Let $U \subset \operatorname{Spec} A$ be an open Zariski subspace given by $A \to A[\frac{1}{f}]$ for $f \in A$. Prove that the base change functor

$$D(A_\square) \to D(A[\frac{1}{f}]_\square)$$

is an open localization. Hint: consider the universal case $\mathbb{Z}[T]_\square \to \mathbb{Z}[T^{\pm 1}]_\square$ and the idempotent $\mathbb{Z}[T]_\square$-algebra $\mathbb{Z}[[T]]$.

(2) Suppose that the cover U_i is of the form $U_i = \operatorname{Spec} A[\frac{1}{f_i}]$ with $(f_1, \ldots, f_n) = A$ generating the unit ideal (we say that the cover is by *standard open Zariski subspaces*). Prove that the family of open localizations $D(A_\square) \to D(A[\frac{1}{f_i}]_\square)$ of the locale $\mathscr{S}(D(A_\square))$ is a cover. Deduce that $D(A_\square)$ has descent for this kind of covers. Hint: this amounts to showing that the intersection of the complement idempotent algebras is zero. First consider the case when A is of finite type over $\mathbb{Z}$, then base change.

(3) Let $U \subset \operatorname{Spec} A$ be an open Zariski subspace. Show that $D(A_\square) \to D(\mathscr{O}(U)_\square)$ is an open localization. Hint: cover U by standard Zariski open subspaces.

(4) Show that the cover $\{U_i\}_i$ has a refinement by an open Zariski cover of the form $\{\operatorname{Spec} A[\frac{1}{f_j}]\}_j$ with $(f_1, \ldots, f_k) = A$ generating the unit ideal. Then deduce that $D(A_\square)$ satisfies descent for the Zariski topology of $\operatorname{Spec} A$. Hint: reduce the question to Exercise 2.3.7 (7).

Exercise 2.3.9. In this exercise we prove a special case of Theorem 2.3.6 for the affine line. Let $f : \mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$.

(1) Show that the base change functor $j^* : D((\mathbb{Z}[T], \mathbb{Z})_\square) \to D(\mathbb{Z}[T]_\square)$ is an open localization with respect to the idempotent algebra $\mathbb{Z}((T^{-1}))$.

(2) Define $f_! : D(\mathbb{Z}[T]_\square) \to D(\mathbb{Z}_\square)$ as $f_! = p_* j_!$ where $j_! : D(\mathbb{Z}[T]_\square) \to D((\mathbb{Z}[T], \mathbb{Z})_\square)$ is the left adjoint of j^*, and $p_* : D((\mathbb{Z}[T], \mathbb{Z})_\square) \to D(\mathbb{Z}_\square)$ is the forgetful functor. Show that for $M \in D(\mathbb{Z}[T]_\square)$ one has

$$f_! M = \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1] \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_\square} M.$$

(3) Compute the right adjoint $f^!$. Show that by taking the residue map

$$\operatorname{res} : \mathbb{Z}((T^{-1}))/\mathbb{Z}[T] \to \mathbb{Z} \cdot T^{-1}$$

one has an isomorphism of functors $D(\mathbb{Z}_\square) \to D(\mathbb{Z}[T]_\square)$

$$f^! \cong f^*[1].$$

3. Analytic Geometry and Analytic Stacks

In the previous lectures we have introduced the categories of condensed sets, condensed abelian groups, solid abelian groups and solid modules over a ring, and we have discussed the category of solid quasi-coherent sheaves over a scheme as well as the statement for a very general Serre duality. In this talk we shall continue with one of the most important notions in condensed mathematics which will generalize the notion of *completion* we used for solid abelian groups or solid modules over a ring, and that of solid quasi-coherent sheaves on a scheme. That is, we will introduce the categories of Analytic Rings and Analytic Stacks.

3.1. **Analytic Rings.** The goal of Analytic Stacks is to create a framework where all different kind of geometries that appear in nature live together and can be compared. But what is an analytic geometry? From the point of view of algebraic geometry two main ingredients are needed: building blocks typically provided by a class of rings, a glueing process that will create more spaces out from rings. In all different analytic geometric frameworks the rings that show up are endowed with a natural topology (Banach, Fréchet, LB, etc.), and the gluing process require a notion of localization which is often achieved via some completed tensor product (I-adically complete, projective or injective tensor products, etc.).

Thanks to the theory of condensed mathematics one can encode the structure of a topological ring simply as a condensed ring; this provides a much better algebraic definition than that of topological ring. What about the completed tensor product? Well, given a condensed/topological ring A there is no a *natural* completed tensor product; even in some situations where there seems to be an *obvious* tensor product like for $\mathbb{Z}_p$, this tensor is a *choice* that is very convenient for applications. Therefore at some point we need to make a choice of a completed tensor product, and this is algebraically encoded in a suitable symmetric monoidal (∞-)category. With no further digressions let us define analytic rings:

Definition 3.1.1. An analytic ring A is the datum of a condensed ring $A^{\triangleright}$ and a full subcategory of A-complete modules $\mathrm{Mod}(A) \subset \mathrm{Mod}(A^{\triangleright})$ satisfying the following conditions:

(1) $\mathrm{Mod}(A)$ is stable under limits, colimits and extensions in $\mathrm{Mod}(A^{\triangleright})$.
(2) For any $N \in \mathrm{Mod}(A)$ and $M \in \mathrm{Mod}(A^{\triangleright})$, for all $i \geq 0$ the condensed Ext-modules

$$\underline{\mathrm{Ext}}^i_{A^{\triangleright}}(M, N)$$

are in $\mathrm{Mod}(A)$.
(3) $A^{\triangleright} \in \mathrm{Mod}(A)$.

We call $\mathrm{Mod}(A)$ the category of A-complete modules.

A map of analytic rings $A \to B$ is a map of condensed rings $A^{\triangleright} \to B^{\triangleright}$ such that any complete B-module $M \in \mathrm{Mod}(B)$ is A-complete when restricted to an $A^{\triangleright}$-module.

Remark 3.1.2. The most general definition of an analytic ring A requires working with higher algebra in the form of animated condensed rings. In this case we would not ask for an abelian category of complete modules but for a stable category of complete modules $D(A) \subset D(A^{\triangleright})$ satisfying some conditions, see [RC24b, Definition 4.1.1]. Nevertheless, complete A-modules are completely determined by the abelian category given by the heart of $D(A)$, see [RC24b, Theorem 4.4.1].

The reason for working with that generality is because the theory itself asks for it: classical Huber's theory of adic spaces ([Hub94]) requires the ring to be *sheafy*, which is a very ad hoc condition. It turns out that by allowing animated rings to fit in the picture one can define adic spaces in a much greater generality, see [RC24a, Section 2.7].

Theorem 3.1.3 ([CS20, Proposition 12.4])**.** *Let A be an analytic ring, then the fully faithful functor* $\mathrm{Mod}(A) \subset \mathrm{Mod}(A^{\triangleright})$ *has a left adjoint* $A \otimes_{\underline{A}} - : \mathrm{Mod}(A^{\triangleright}) \to \mathrm{Mod}(A)$ *(understood as the completion functor). Moreover, the category* $\mathrm{Mod}(A)$ *is abelian and has a unique symmetric monoidal structure, denoted as* $\otimes_A$*, making* $A \otimes_{\underline{A}} -$ *symmetric monoidal (we think of* $\otimes_A$ *as the completed tensor product).*

Example 3.1.4. (1) Given a condensed ring $A^{\triangleright}$ one can construct the *trivial analytic ring structure* $\underline{A}^{\triangleright}$ whose category of complete modules are all condensed modules.
(2) The pair $\mathbb{Z}_{\square} = (\mathbb{Z}, \mathsf{Solid})$ consisting on the integers and the category of solid abelian groups forms an analytic ring; this is the content of Theorem 2.1.4.
(3) More generally, for a discrete ring A the pair $A_{\square} = (A, \mathrm{Mod}(A_{\square}))$ consisting on the ring A and the category of solid A-modules forms an analytic ring; this is the content of Theorem 2.2.6.

Exercise 3.1.5. Let A be an analytic ring and let $B^{\triangleright}$ be an $A^{\triangleright}$-algebra which is A-complete as a module. Consider $\mathscr{C} \subset \mathrm{Mod}(B^{\triangleright})$ the full subcategory of $B^{\triangleright}$-modules whose underlying $A^{\triangleright}$-module is complete. Show that the pair $B^{\triangleright}_{A/} := (B^{\triangleright}, \mathscr{C})$ is an analytic ring. We call $B^{\triangleright}_{A/}$ the *induced analytic ring structure* from A to $B^{\triangleright}$.

Remark 3.1.6. The initial definition of analytic rings used a different approach involving the so called *measures on profinite sets*. The heuristics behind is the following: let A be an analytic ring. Given S a light profinite set the free A-module

$$A[S] = A \otimes_{A^{\triangleright}} A^{\triangleright}[S]$$

can be considered as a space of "A-valued measures on S". Then, a complete A-module M is a condensed $A^{\triangleright}$-module for which for all profinite set S and any map

$$f : S \to M$$

one can "integrate" the map f along any measure $\mu \in A[S]$ in a continuous, unique and functorial way. Concretely this means that the natural map

$$R\,\underline{\mathrm{Hom}}_A(A^{\triangleright}[S], M) \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}_A(A[S], M)$$

is an equivalence.

This perspective on measures fits very well with the solid formalism: given a discrete ring A of finite type over $\mathbb{Z}$ and S a profinite set one has

$$A_{\square}[S] = \underline{\mathrm{Hom}}_A(C(S, A), A)$$

where $C(S, A)$ is the space of locally constant functions from S to A. Thus, $A_{\square}[S]$ is the space of A-valued measures for the discrete topology on A.

Example 3.1.7. Previously we have discussed solid tensor products which are great to do non-archimedean analysis, eg. by Exercise 2.1.22 solid tensor products of Banach spaces are Banach spaces. There are other very important analytic ring structures that interpolate between p-adics and real numbers called the *gaseous* and *liquid* analytic ring structures. We will not discuss how these analytic rings are constructed (as they take a while and the construction of the liquid rings is really involved), but instead let us mention some features:

et $\mathbb{R}^{\mathrm{gas}}$ be the ring of real numbers endowed with the gaseous analytic ring structure, similarly (for $0 < p \leq 1$) some fix parameter let $\mathbb{R}_{<p}$ be the real numbers endowed with the p-liquid analytic ring structure.

i. Let F_1 and F_2 be nuclear Fréchet spaces, then $F_1 \otimes_{\mathbb{R}_{<p}} F_2$ is a nuclear Fréchet space equal to the projective tensor product of F_1 and F_2. If F_1 and F_2 are in addition nuclear Fréchet spaces which can be written as a limit of Banach spaces with trace class maps of quasi-exponential decay, then $F_1 \otimes_{\mathbb{R}^{\mathrm{gas}}} F_2$ is also the projective tensor product.

ii. As a more concrete example, let $\mathscr{O}(\mathbb{D}_{\mathbb{C}})$ be the space of holomorphic functions on the open unit disc. Then

$$\mathscr{O}(\mathbb{D}_{\mathbb{C}}) \otimes_{\mathbb{R}^{\mathrm{gas}}} \mathscr{O}(\mathbb{D}_{\mathbb{C}}) = \mathscr{O}(\mathbb{D}_{\mathbb{C}}) \otimes_{\mathbb{R}_{<p}} \mathscr{O}(\mathbb{D}_{\mathbb{C}}) = \mathscr{O}(\mathbb{D}^2_{\mathbb{C}}).$$

This roughly means that the gaseous or liquid tensor product are good frameworks to do complex analytic geometry.

iii. On the other hand, let M and N be compact smooth manifolds. Then smooth functions behaved as expected for the liquid tensor product:

$$C^{\infty}(M, \mathbb{R}) \otimes_{\mathbb{R}_{<p}} C^{\infty}(N, \mathbb{R}) = C^{\infty}(N \times M, \mathbb{R}).$$

However, this computation fails for the gaseous tensor product. This roughly says that one needs the liquid tensor product for a good framework for smooth manifolds.

iv. Tensor products of Banach spaces is in general **not** a Banach space for the gaseous or liquid tensor product. Actually, a natural candidate to define an analytic ring over $\mathbb{R}$ is by taking Radon measures on profinite sets, in [CS20, Lecture IV] it is shown that this is **not** possible to get an analytic ring out of that.

Having defined the building blocks in our theory one has to figure out the way to localize and glue. The idea is to define the most general topology in analytic rings so that essentially all existent topologies in algebraic and analytic geometry are included. Defining this topology requires new technology that goes beyond the scope of these notes, namely the theory of abstract six functor formalisms [Man22b, Sch23, HM24]. Hence, let us informally mention what are the main features of this topology:

i. In AnRing there is a six functor formalisms for the derived categories of quasi-coherent sheaves $A \mapsto D(A)$. There is a class of special maps of analytic rings $f : A \to B$ (that could be called compactifiable) for which $f_!$ is defined.
ii. The Grothendieck topology on AnRing is called the *!-topology*, a map of analytic rings $f_! : A \to B$ is a !-cover if the category of complete modules $D(A)$ satisfies !-descent.
iii. Asking for a map to be a !-cover is very strong. Indeed, if $f : A \to B$ has !-descent then it satisfies usual descent along pullback maps. Furthermore, it satisfies descent in the 2-category of presentable $D(A)$-linear categories.
iv. For a morphism $A \to B^{\triangleright}_{A/}$ with the induced analytic ring structure, being a !-cover is equivalent to being descendable in the sense of Mathew [Mat16].

Exercise 3.1.8. Let us provide an example of a !-cover which is related with Beauville-Laszlo gluing. Let A be a discrete ring of finite type over $\mathbb{Z}$ and let $a \in A$ be an element. Consider the derived category $D((A, \mathbb{Z})_{\square})$ of A-modules on solid abelian groups. Consider the solid algebras $A[\frac{1}{a}]$ and $\widehat{A} = \varprojlim_n A/a^n$ given by inverting a and completing along (a).

(1) Show that $\widehat{A}$ is an idempotent algebra in $D((A, \mathbb{Z})_{\square})$.
(2) Show that there is a short exact sequence

$$0 \to A \to A[\frac{1}{a}] \oplus \widehat{A} \to \widehat{A}[\frac{1}{a}] \to 0.$$

Prove that the locale $\mathscr{S}(D((A, \mathbb{Z})_{\square}))$ is also the union of the closed subspaces attached to the algebras $A[\frac{1}{a}]$ and $\widehat{A}$. Hint: see Exercise 2.3.7 (8). Deduce that we have a cartesian diagram of ∞-categories

$$\begin{array}{ccc} D((A,\mathbb{Z})_{\square}) & \longrightarrow & D((A[\frac{1}{a}],\mathbb{Z})_{\square}) \\ \downarrow & & \downarrow \\ D((\widehat{A},\mathbb{Z})_{\square}) & \longrightarrow & D((\widehat{A}[\frac{1}{a}],\mathbb{Z})_{\square}). \end{array}$$

Since the algebra $A[\frac{1}{a}] \times \widehat{A}$ is descendable in $D((A, \mathbb{Z})_{\square})$ the map of analytic rings

$$(A, \mathbb{Z})_{\square} \to (A[\frac{1}{a}] \times \widehat{A}, \mathbb{Z})_{\square}$$

is a !-cover. In general, a finite family of idempotent algebras $A \to A^{\triangleright}_{i,A/}$ with the induced analytic ring structure whose union of closed subspaces is the total space of the locale is a !-cover, cf. Exercise 2.3.7 (8). Similarly, a family of maps $A \to A_i$ of analytic rings that give rise open immersions $D(A) \to D(A_i)$ and that cover the total space of the locale $\mathscr{S}(D(A))$ is also a !-cover.

3.2. **Analytic Stacks.** We have discussed the building blocks for analytic geometry, aka analytic rings, and mentioned that they carry a very general Grothendieck topology that is helpful to make many desired localizations. We can give the following (partial) definition of the category of analytic stacks:

Definition 3.2.1. Let $\mathscr{C} = \mathsf{AnRing}^{\mathrm{op}}$ be the opposite category of the ∞-category of analytic rings. The category of analytic stacks is the category of anima-valued sheaves $\mathrm{Sh}_!(\mathscr{C}, \mathsf{Ani})$ of $\mathscr{C}$ endowed with the !-topology. We let AnStk be the ∞-category of analytic stacks.

Remark 3.2.2. In order to capture the correct tensor products of analytic rings in general we are force to work fully derived. However, there are many cases (eg. schemes) where one can look at some subcategories of rings where the non derived and derived base changes agree. The reader that feels uncomfortable with ∞-categories can get the picture that analytic stacks are similar to classical stacks in algebraic geometry, except that now we are using analytic rings as building blocks, and the !-topology instead of the flat topology, eg. that we are taking quotients of affine/affinoid spaces by equivalence relations which are !-covers.

Remark 3.2.3. One might try to built up a class of analytic stacks which look pretty much like algebraic stacks, i.e. that are quotients of schemes by smooth equivalence relations. However, different geometric theories have different meanings for smooth maps, and a priori it is not clear how one can generalize those notions. Nevertheless, by using the theory of six functor formalisms one can define a suitable class of *!-able analytic stacks* which are roughly speaking quotients of disjoint unions of affinoid rings by *very nice* equivalence relations (where "very nice" is related with !-descend for quasi-coherent sheaves).

By Yoneda's lemma we have a fully faithful embedding

$$\mathsf{AnRing}^{\mathrm{op}} \hookrightarrow \mathsf{AnStk} \tag{3.1}$$

Given an analytic ring A we write $\mathrm{AnSpec}\, A \in \mathsf{AnStk}$ for the analytic stack it represents, and call it the *analytic spectrum* of A. An object in the essential image of (3.1) is called an *affinoid analytic stack*.

Remark 3.2.4. Classically in algebraic geometry, given a ring R one defines the spectrum $|\operatorname{Spec} R|$ to be the topological space of prime ideals endowed with the Zariski topology. One then upgrades $\operatorname{Spec} R$ to a scheme that is written in the same way. However, one can introduce the scheme $\operatorname{Spec} R$ formally as the (pre)sheaf on the category of commutative rings represented by R, and then think of the topological space $|\operatorname{Spec} R|$ as a *construction* attached to this sheaf. In the theory of analytic stacks there is not a unique space that one could attach to $\mathrm{AnSpec}\, A$ (the most general topological-like object being the smashing spectrum $\mathscr{S}(D(A))$ of its derived category of complete modules), so the notation "$\mathrm{AnSpec}\, A$" is essentially formal and only refers to the sheaf A represents.

The definition of the category of analytic stacks was made so that one can construct categories of quasi-coherent sheaves:

Definition 3.2.5. Let $X \in \mathsf{AnStk}$ be an analytic stack, its ∞-derived category of quasi-coherent sheaves is given by

$$D(X) = \varprojlim_{\mathrm{AnSpec}\, A \to X} D(A)$$

where the index of the limit is the slice category $\mathrm{AnRing}^{\mathrm{op}}_{/X}$ of affinoid analytic stacks over X. Informally, the datum of a quasi-coherent sheaf on X is the datum of an A-module M_A for any map $\mathrm{AnSpec}\, A \to X$, and for any diagram

$$\begin{array}{ccc} \mathrm{AnSpec}\, B & \longrightarrow & \mathrm{AnSpec}\, A \\ & \searrow & \downarrow \\ & & X \end{array}$$

an isomorphism $M_A \otimes_A B \cong M_B$ subject to higher coherences.

Remark 3.2.6. Different from what happens for usual categories of quasi-coherent sheaves for schemes, the quasi-coherent category $D(X)$ of an analytic stack does not have a natural t-structure. The reason for this apparent defect is that !-covers of analytic rings **are not** flat. However, in many situations appearing in practice one can construct subcategories in $D(X)$ that can be endowed with t-structures (like quasi-coherent sheaves in classical schemes, or D-modules in de Rham stacks). We do not see this lack of a t-structure as a problem in the theory of analytic stacks since it is not important for many constructions.

The theory of analytic stacks is very powerful since it allows us to construct a lot of crazy stacks that can involve objects of very different nature. However, it requires some time and effort to learn. In the next section we will give some examples of analytic stacks that are related with more classical analytic, topological or algebraic objects.

3.3. **Examples of Analytic Stacks.**

3.3.1. *Betti stacks.* A first important class of analytic stacks are those constructed from condensed sets; they are called *Betti stacks* and construct as follows:

First, we construct a functor $(-)_{\mathrm{Betti}}\mathsf{Prof} \to \mathsf{AnStk}$ from the category of light profinite sets to the category of analytic stacks. It is given by mapping a light profinite set S to $\mathrm{AnSpec}\, C(S,\mathbb{Z})$ where $C(S,\mathbb{Z})$ is the space of continuous (eq. locally constant) functions from S to $\mathbb{Z}$. It turns out that given a surjective map $S' \to S$ of profinite sets the morphism of rings

$$C(S,\mathbb{Z}) \to C(S',\mathbb{Z})$$

is faithfully flat, and being countably generated it is also descendable [Mat16, Proposition 3.31] (and so of !-descent). Thanks to this fact, we can descend the formation of $S \mapsto S_{\mathrm{Betti}}$ to condensed sets (more generally to condensed anima) and get a functor

$$(-)_{\mathrm{Betti}} : \mathsf{CondSet} \to \mathsf{AnStk}$$

which one can show is fully faithful. By design, given $T \in \mathsf{CondSet}$ the derived category of quasi-coherent sheaves of its Betti stack $D(T_{\mathrm{Betti}})$ contains fully faithfully the derived category $D(\underline{T})$ of Definition 1.3.5. For X a locally compact Hausdorff space $D(X_{\mathrm{Betti}})$ is the left completion of the derived category of condensed abelian sheaves on X; we can think of X_{Betti} as its realization as *topological space*.

3.3.2. *Schemes and algebraic stacks.* Given a classical (animated) commutative ring R we can consider the analytic ring $R^{\triangleright}$ consisting on $\underline{R}$ endowed with the full subcategory of condensed $\underline{R}$-modules. By endowing the category of affine schemes $\mathsf{Ring}^{\mathrm{op}}$ with the fpqc topology[1] one can construct the category of algebraic stacks $\mathsf{AlgStk} := \mathrm{Sh}_{\mathrm{fpqc}}(\mathsf{Ring}^{\mathrm{op}}, \mathsf{Ani})$. Then, the natural inclusion

$$\mathsf{Ring}^{\mathrm{op}} \hookrightarrow \mathsf{AnRing} \hookrightarrow \mathsf{AnStk}$$

will extend to a functor

$$(-)^{\triangleright} : \mathsf{AlgStk} \to \mathsf{AnStk}$$

from algebraic stacks to analytic stacks. It is not clear to the author whether this functor is fully-faithful (in [CS24] it is mentioned that at least for projective schemes the functor is fully faithful, it is likely that it is fully faithful in a large class of algebraic stacks). However, given an algebraic stack $X \in \mathsf{AlgStk}$, eg. a scheme, the category $D(X^{\triangleright})$ is essentially the same as the category of quasi-coherent sheaves of X, except that it is enriched on condensed abelian groups. We then have a fully faithful embedding

$$D(X) \hookrightarrow D(X^{\triangleright})$$

from the classical derived category of quasi-coherent sheaves to the derived category of $X^{\triangleright}$.

[1]Actually, one would need to consider a variant of the fpqc topology generated by *countably presented* fpqc covers.

3.3.3. *Complex varieties.* Complex varieties also have a natural realization in analytic stacks; this is essentially done in [CS22]. The idea behind is to rebuilt complex analytic spaces (and a derived generalization) not from holomorphic functions in open polydiscs $\mathbb{D}^d_{\mathbb{C}}$ but from overconvergent functions on closed polydiscs $\overline{\mathbb{D}}^d_{\mathbb{C}}$.

To realize this construction we need to work with a completed tensor product over $\mathbb{C}$, namely, we can take either the gaseous tensor product $\mathbb{C}^{\mathrm{gas}}$ [CS24] or the liquid tensor product $\mathbb{C}_{<p}$ [CS20] (depending on a parameter $0 < p \leq 1$). One then defines a category of affinoid $\mathbb{C}$-algebras (always endowed with the induced analytic structure from $\mathbb{C}^{\mathrm{gas}}$ or $\mathbb{C}_{<p}$) which are essentially quotients of the algebras $\mathscr{O}(\overline{\mathbb{D}}^n_{\mathbb{C}})$ of overconvergent holomorphic functions in the polydisc (it is possible to take a more generall class of algebras coming from Stein spaces but to reconstruct complex analytic manifolds the former are enough). The use of the gaseous or liquid tensor products is important for the following expected computation to hold:

$$\mathscr{O}(\overline{\mathbb{D}}^n_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathscr{O}(\overline{\mathbb{D}}^m_{\mathbb{C}}) = \mathscr{O}(\overline{\mathbb{D}}^{n+m}_{\mathbb{C}}).$$

This guarantees that the fibre products of complex analytic spaces seen as analytic stacks match with the expected ones.

Given an affinoid algebra A one has an attached Berkovich spectrum $\mathcal{M}(A)$ which is nothing but the condensed algebra homomorphisms $A \to \mathbb{C}$ (it agrees with the classical Berkovich spectrum), and $\mathcal{M}(A)$ is a compact Hausdorff space. Furthermore, one can construct a morphism of analytic stacks

$$\operatorname{AnSpec} A \to \mathcal{M}(A)_{\mathrm{Betti}} \tag{3.2}$$

which basically says that the category $D(A)$ of complete A-modules satisfies descent for the topology of $\mathcal{M}(A)$.

Let $\mathsf{Aff}_{\mathbb{C}}$ be the category of affinoid $\mathbb{C}$-algebras, we can endow its opposite category with the analytic topology arising from the topology on Berkovich spaces, and define complex analytic stacks $\mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}}, \mathsf{Ani})$. Complex analytic spaces are then complex analytic stacks that are, locally for the analytic topology, representable by affinoid algebras. The fully faithful embedding

$$\mathsf{Aff}^{\mathrm{op}}_{\mathbb{C}} \hookrightarrow \mathsf{AnRing}^{\mathrm{op}} \hookrightarrow \mathsf{AnStk}$$

extends to a colimit preserving functor

$$(-)^{\mathrm{an}} : \mathrm{Sh}_{\mathrm{an}}(\mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}}, \mathsf{Ani}) \to \mathsf{AnStk}$$

that gives rise the realization of conmplex analytic varieties as a analytic stacks. We highlight that because of (3.2), given X a complex analytic variety, there is a natural map

$$X^{\mathrm{an}} \to |X|_{\mathrm{Betti}}$$

where $|X|_{\mathrm{Betti}}$ is the underlying topological spaces of $\mathbb{C}$-valued points of X.

Remark 3.3.1. The previous construction can be also made over $\mathbb{Q}_p$ or over a global gaseous or liquid base, obtaining a realization of †-varieties over different field-bases as analytic stacks.

3.3.4. *Rigid varieties and adic spaces.* Recall that adic spaces are built up from the theory of Huber pairs [Hub94]. An Huber pair is a pair (R, R^+) consisting on an f-adic ring R and an integrally closed open subring $R^+ \subset R$ of bounded elements. Given an Huber ring (R, R^+) there is a natural realization as analytic ring which was first defined in [CS19]. The idea is the following: we send (R, R^+) to the solid ring $(R, R^+)_{\square}$ with underlying ring R and whose category of complete modules are condensed R-modules which are a-solid for all $a \in R^+$. By [And21, Proposition 3.34].

Furthermore, [And21, Theorem 4.1] (plus a small argument) shows that open analytic localizations of $\mathrm{Spa}(R, R^+)$ give rise to !-able covers of the analytic ring $(R, R^+)_{\square}$. Let $\mathsf{AffPair}$ be the category of affinoid Huber pairs, then the fully faithful inclusion $\mathsf{AffPair}^{\mathrm{op}} \hookrightarrow \mathsf{AnStk}$ will extend to a functor

$$(-)^{\mathrm{an}} : \mathsf{AdicSpaces} \to \mathsf{AnStk}$$

from the category of adic spaces into analytic stacks. Given an adic space X we also have a map $X^{\mathrm{an}} \to \mathcal{M}(X)_{\mathrm{Betti}}$ from the realization of X as analytic stack to the Betti realization of its underlying Berkovivh space. The category $D(X^{\mathrm{an}})$ is the ∞-category of solid quasi-coherent sheaves on X.

3.3.5. *Real and p-adic analytic manifolds.* Giving a real analytic or p-adic manifold X one has a structure sheaf of real analytic or $\mathbb{Q}_p$-locally analytic functions on X. From this datum, using the gaseous tensor product for example, we obtain a realization X^{la} of X as real anlaytic or $\mathbb{Q}_p$-locally analytic variety into analytic stacks. If K is a real Lie group then quasi-coherent sheaves on the classifying stack BK^{la} encodes a representation category of real analytic representations which makes an important part of the geometrization of the real local Langlands correspondence [Sch24]. Similarly, if G is a p-adic Lie group then quasi-coherent sheaves on BG^{la} are the same as locally analytic representations of G [RJRC22, RJRC23]; these stacks also show up naturally in the geometrization of the p-adic locally analytic Langlands correspondence.

Finally, given X a complex analytic manifold, we have a natural map

$$X^{la} \to X^{\mathrm{an}}$$

from the real analytic to the complex analytic incarnations of X, arising from the inclusion of holomorphic functions into real analytic functions.

3.3.6. *Analytic GAGA.* Let X be an algebraic variety over $\mathbb{C}$. We have then different stacks that we can attach to X (we work over $\mathbb{C}^{\mathrm{gas}}$ or $\mathbb{C}_{<p}$):

- X_{top} the topological manifold given by the analytic spectrum of the continuous functions $C(X(\mathbb{C}), \mathbb{C})$.
- X^{∞} the differentiable manifold given by the analytic spectrum of the smooth functions $C^{\infty}(\mathbb{X}(\mathbb{C}), \mathbb{C})$.
- X^{la} the real analytic manifold given by the analytic spectrum of the real analytic functions $C^{\omega}(\mathbb{X}(\mathbb{C}), \mathbb{C})$.
- The analytic complex manifold X^{an}.
- The (base change to $\mathbb{C}^{\mathrm{gas}}$ or $\mathbb{C}_{<p}$ of the) algebraic variety $X^{\triangleright}$.
- The (base change to $\mathbb{C}^{\mathrm{gas}}$ or $\mathbb{C}_{<p}$ of the) Betti realization $X(\mathbb{C})_{\mathrm{Betti}}$.

We have maps of analytic stacks (over $\mathbb{C}^{\mathrm{gas}}$ or $\mathbb{C}_{<p}$)

$$X_{\mathrm{top}} \to X^{\infty} \to X^{la} \to X^{\mathrm{an}} \to X^{\triangleright}$$

and

$$X^{\mathrm{an}} \to X(\mathbb{C})_{\mathrm{Betti}}.$$

Then a geometric incarnation of GAGA is the fact that for X a proper scheme over $\mathbb{C}$ the map of analytic stacks

$$X^{\mathrm{an}} \xrightarrow{\sim} X^{\triangleright}$$

is an equivalence. A similar statement hold for rigid varieties, or any other kind of situation where GAGA applies. Deducing classical GAGA from this still requires some work, see [CS22], but the geometric statement captures the real essence of what GAGA is about.

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