NOTES ON SOLID GEOMETRY

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Abstract. These are notes of a seminar held in Columbia university during the Spring and Fall of 2024 about the new theory of analytic stacks of Clausen and Scholze. The seminar is inspired from the Lecture Series of Analytic Stacks. All results must be attributed to Clausen and Scholze and any mistake or misconception is totally due to the author.

Contents

1. Introduction

Different geometric theories appear all across mathematics: differentiable manifolds, complex and real analytic varieties, rigid analytic spaces, adic spaces, Berkovich spaces, algebraic varieties and schemes, formal schemes, etc. The aim of "analytic stacks" is to define a general ecosystem where the previous (and many more!) "theories of analytic and algebraic geometries" cohabit and interact each other. To motivate the distribution of future talks let us make explicit the obstructions that mathematicians have face all over the years when dealing with analytic geometry, and how condensed mathematics and analytic stacks have solved these issues.

1.1. **Light condensed sets.** The building blocks in theories such as algebraic varieties or schemes consist simply of commutative rings satisfying some additional algebraic properties. This leads to a pleasant treatment of geometry that is studied in purely algebraic terms. However, in other theories such as differentiable manifolds and complex or rigid analytic varieties, the building blocks turn out to be some sort of topological rings, more often Banach or Fréchet rings. Then, any general form of "analytic geometry" that inherits a similar formalism as algebraic geometry must be built up over an algebraic theory of "topological rings". However, history has shown that the datum of a topology does not mixes very well with that of an algebraic structure. A very simple and clever solution to this is provided by condensed mathematics [CS19], where "topology" is changed by the topos of (light) condensed sets. Therefore, our first replacement for topological "*preferred algebraic structure*" (eg. ring/module/abelian group/monoid) will be condensed "*preferred algebraic structure*".

The idea behind condensed mathematics follows the philosophy of Grothendieck saying that a space X must be studied by looking at maps $Y \to X$ from some "test objects" Y. For this approach to be useful, one needs to choose the "test objects" wisely. In our situation, we want to study (reasonable) topological spaces, and a first class of reasonable topological spaces are compact Hausdorff spaces. It turns out that compact Hausdorff spaces can be reconstructed from a certain class of "very acyclic" spaces. Concretely, let Prof be the category of profinite sets/totally disconnected compact Hausdorff spaces. We endow Prof with the Grothendieck topology whose covers are given by finitely many jointly surjective maps. As justification for this choice, recall that any surjective map of compact Hausdorff spaces is a quotient map, and that any compact Hausdorff space X admits a surjection from a profinite set. For instance, the closed interval $[0, 1]$ admits a surjective map from the Cantor set $\prod_{\mathbb{N}}\{0, 1\} \to [0, 1]$ by sending a sequence (a_n) to the real number written in binary decimals

$$(a_n) \mapsto 0.a_1a_2a_2 \cdots .$$

Definition 1.1.1. A condensed set is a sheaf $T : \mathrm{Prof}^{\mathrm{op}} \to \mathrm{Set}$ (modulo some set-theoretical technicalities i.e. accessible), we let CondSet denote the category of condensed sets. For X a (reasonable) topological space (eg. Hausdorff), we define its condensification $\underline{X} \in \mathrm{CondSet}$ by taking

$$\underline{X}(S) = \mathrm{Map}(S, X)$$

the space of continuous maps from S to X, with $S \in \mathrm{Prof}$.

Most of the spaces we care of in topology (such as countably generated CW complexes), geometry (eg. manifolds), and analysis (eg. Banach, Fréchet spaces) are endowed with a topology for which understanding converging sequences is often enough. More precisely, the most interesting topological spaces are (locally) metrizable. Thus, a good balance in condensed mathematics between capturing all the relevant information and avoiding unnecessary technicalities is given by light condensed sets:

Definition 1.1.2. A light profinite set is a metrizable profinite set, we $\mathrm{Prof}^{\mathrm{light}}$ be the category of light profinite sets. A light condensed set is a sheaf $T : \mathrm{Prof}^{\mathrm{light,op}} \to \mathrm{Set}$, we let $\mathrm{CondSet}^{\mathrm{light}}$ denote the category of light condensed sets.

Finally, for any algebraic structure $\mathscr{C}$ (aka. a category with small limits and colimits), its condensification $\mathrm{Cond}(\mathscr{C})$ is the category of sheaves $T : \mathrm{Prof}^{\mathrm{light,op}} \to \mathscr{C}$ from light profinite sets in $\mathscr{C}$. For example, we can talk about (light) condensed abelian groups, rings, monoids, etc. The notion of light condensed "*preferred algebraic structure*" is the replacement we shall use for its topological analogue.

1.2. **Analytic rings.** As it was mentioned before, part of the datum of the building blocks in a general theory of analytic geometry must involve some kind of topological (aka condensed) ring. On the other hand, the most fundamental invariant of a space X in both analytic or algebraic geometry is its category of (quasi-)coherent sheaves $\mathrm{QCoh}(X)$. In classical algebraic geometry this category is obtained by gluing, using "Zariski descent", the category of modules $\mathrm{Mod}(A)$ of commutative rings A. However, in the case of complex and rigid geometries, the best that one can (classically) do in a systematic and algebraic manner is to built up the category of coherent modules $\mathrm{Coh}(X)$, imposing in this way finiteness conditions to the sheaves living over X. In particular, for a general morphism $f : X \to Y$ of rigid or complex analytic spaces, the sheaf $f_*\mathscr{O}_X$ does not belong to the category attached to X. On the other hand, even though condensed rings are some kind of topological rings, in analytic geometries we often want to have some kind of "complete tensor product" and a category of "complete modules". It turns out that if A and B are two condensed rings, then the underlying ring of $A \otimes_{\mathbb{Z}} B$ is just the algebraic tensor $A(*) \otimes_{\mathbb{Z}} B(*)$, proving that we still need to do something else.

The notion of analytic ring appears as a solution to the previous problematics. The datum of an analytic ring A consists of a condensed ring $A^{\triangleright}$ and a stable ∞-category $\mathscr{D}(A)$ of "complete A-modules". Before enumerating the features of $\mathscr{D}(A)$, let us do a brief detour explaining this jump from an abelian category of modules to a stable ∞-category: in classical algebraic geometry, the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves is endowed with a symmetric tensor product $\otimes_{\mathscr{O}_X}$. Within this tensor product one can construct fiber products $X \times_Y Z$ of (affine) schemes by simply taking the (affine) scheme represented by the tensor product of rings. However, when dealing with cohomological invariants of algebraic varieties, it is natural to enter the world of derived categories. In this realm the "correct fiber product" $X \times_Y Z$ is not longer constructed using the "abelian" tensor product of rings but instead the "derived tensor product". Thanks to the current status of higher category theory and higher algebra, eg. [Lur09, Lur17, Lur18], we have nowadays the categorical tools to develop theories of "derived algebraic geometries" as in [Lur04, Lur18].

In the former theory of analytic geometry, classical abelian or triangulated categories of quasi-coherent sheaves are not enough to obtain descent and glue to more general spaces (a reason is the lack of "complete" flatness even for some simple maps such as open immersions of rigid or complex analytic spaces). Instead, stable ∞-categories are perfectly suited for these purposes. As consequence of the previous explanation, the general theory of analytic rings depends in higher categorical foundations (eg. the underlying condensed ring $A^{\triangleright}$ should be an animated or a condensed $\mathbb{E}_{\infty}$-ring), even though the most fundamental examples still can be explained in the world of abelian categories. For the reader that is not comfortable with the language of higher category theory, we recommend to consider $\mathscr{D}(A)$ as a classical derived category in a first approach, and accept some features of ∞-derived categories for granted such as the existence of arbitrary (small) limits of ∞-categories [Lur09, §3.3.3], or the adjoint functor theorem [Lur09, Corollary 5.5.2.9].

Going back to the category $\mathscr{D}(A)$, it ought satisfy the following properties:

(1) It should be a full subcategory $\mathscr{D}(A) \subset \mathscr{D}(A^{\triangleright})$ of the derived ∞-category of condensed $A^{\triangleright}$-modules stable under all limits and colimits, and "tensored over condensed abelian groups". This are the basic requirements for doing homological algebra over A.
(2) There is a "completion functor" $A \otimes_{A^{\triangleright}} - : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}(A)$, left adjoint to the natural inclusion (note that we have dropped derived decorations in the tensor). Moreover, $\mathscr{D}(A)$

can be uniquely promoted to a symmetric monoidal category such that $A \otimes_{A^{\triangleright}} -$ is a symmetric monoidal functor. Similarly as for schemes, we require our category of modules to be endowed with a "complete tensor product" that will generalize "complete tensor products" in classical theories of analytic geometries.

(3) The completion functor $A \otimes_{A^{\triangleright}} -$ should preserve connective objects: $A \otimes_{A^{\triangleright}} - : \mathscr{D}(A^{\triangleright})_{\geq 0} \to \mathscr{D}(A^{\triangleright})_{\geq 0}$. This will endow $\mathscr{D}(A)$ with a t-structure arising from condensed $A^{\triangleright}$-modules.

(4) We have $A^{\triangleright} \in \mathscr{D}(A)$ (we want our topological ring to be complete!).

Definition 1.2.1. An analytic ring A is a pair $(A^{\triangleright}, \mathscr{D}(A))$ consisting on a light condensed animated ring $A^{\triangleright}$, and a full subcategory $\mathscr{D}(A) \subset \mathscr{D}(A^{\triangleright})$ of "complete modules" satisfying properties (1)-(4) above. A morphism of analytic rings $f : A \to B$ is a morphism of condensed rings $A^{\triangleright} \to B^{\triangleright}$ such that the forgetful functor $f_* : \mathscr{D}(B^{\triangleright}) \to \mathscr{D}(A^{\triangleright})$ sends $\mathscr{D}(B)$ to $\mathscr{D}(A)$. We let AnRing denote the ∞-category of analytic rings.

It turns out that AnRing a is a presentable ∞-category (cf. [Lur09, §5.5] for the notion of presentability), in particular it admits all (small) colimits (cf. [CS20, Proposition 12.12] and [Man22, Proposition 2.3.15]). Analytic rings shall be the bulding blocks in the theory of analytic stacks.

1.3. **Analytic stacks.** Let Ring be the category of rings. Schemes are constructed out from Ring by gluing using the Zariski topology. In particular, a scheme can be seen as an object in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Ring}^{\mathrm{op}}, \mathrm{Set})$, i.e. a sheaf for the Zariski topology in the opposite category of rings, aka, affine schemes. Similarly algebraic spaces (resp. Artin stacks) are obtained by "gluing affine schemes" along étale or smooth maps, they then define sheaves in more refined Grothendieck topologies such as the étale or flat topologies. Moreover, when defining stacks in derived algebraic geometry [Lur04], it is mandatory to not just consider functors with values in sets but in anima Ani (aka. ∞-groupoids or spaces).

For the theory of analytic stacks we want to define a suitable Grothendieck topology G on AnRing such that "analytic stacks" are given by (hyper)sheaves

$$\mathrm{AnStack} = \mathrm{Sh}_G(\mathrm{AnRing}^{\mathrm{op}}, \mathrm{Ani}).$$

The question that arises is which Grothendieck topology should we consider? Well, by definition analytic rings **are not just** its underlying condensed ring but its category of modules. Indeed, an analytic ring is (essentially) completely determined by its category of modules! Thus, whatever Grotendieck topology we choose, the functor $A \mapsto \mathscr{D}(A)$ should certainly satisfy descent. On the other hand, we want a refined enough Grothendieck that explains already existing "identifications" from classical analytic geometries:

Let $\mathbb{Q}_p$ be the field of p-adic numbers, and consider the projective space $\mathbb{P}^1_{\mathbb{Q}_p}$. There are different ways to construct $\mathbb{P}^1_{\mathbb{Q}_p}$. First, we have the algebraic geometry manner that glues the (spectrum of the) rings $\mathbb{Q}_p[T]$ and $\mathbb{Q}_p[T^{-1}]$ along the intersection $\mathbb{Q}_p[T^{\pm 1}]$. On the other hand, we have rigid geometry and we can construct $\mathbb{P}^1_{\mathbb{Q}_p}$ by gluing the (adic spectrum of the) Tate algebras $\mathbb{Q}_p\langle T\rangle$ and $\mathbb{Q}_p\langle T^{-1}\rangle$ along the intersection $\mathbb{Q}_p\langle T^{\pm 1}\rangle$. Thus, we want the theory of analytic stacks to be able to identify these both constructions of $\mathbb{P}^1_{\mathbb{Q}_p}$ as the same space, getting as a result a geometric version of GAGA theorems.

In later talks we shall introduce the formal definition of the Grothendieck topology used for defining analytic stacks. A key tool in its definition will be the abstract theory of six functor formalisms built for analytic rings.

1.4. **Examples.** During the introduction of light condensed sets, analytic rings and analytic stacks, we shall study in more detail some examples arising from algebraic geometry and the theory of adic spaces (solid theory). We will just shortly mention the existence and some features of archimedean and global examples of analytic rings (liquid and gaseous theory).

Solid abelian groups. Let $\mathrm{CondAb}^{\mathrm{light}}$ denote the category of light condensed abelian groups. We shall define the subcategory of (light) solid abelian groups $\mathrm{Solid} \subset \mathrm{CondAb}^{\mathrm{light}}$ by imposing a condition extracted from the idea that "converging sequences in non-archimedean analysis are precisely the null sequences". The category of solid abelian groups is endowed with a tensor product that we denote by $\otimes_{\square}$, it has $\mathbb{Z}$ as unit, and so it defines and analytic ring $\mathbb{Z}_{\square}$ that we call the "solid integers". The category Solid has a compact projective generator $\prod_{\mathbb{N}} \mathbb{Z}$ that is flat for $\otimes_{\square}$, and satisfies

$$\prod_{I} \mathbb{Z} \otimes_{\square} \prod_{J} \mathbb{Z} = \prod_{I \times J} \mathbb{Z}$$

for countable sets I, J. This category is completely disjoint from archimedean analysis, namely, the solidification of $\mathbb{R}$ is just 0. Examples of solid abelian groups are discrete groups, p-adically complete modules, $\mathbb{Q}_p$-Banach and Fréchet spaces, etc. It also holds that (most) of the completed tensor products appearing in non-arhimidean geometry coincide with $\otimes_{\square}$ (eg. p-complete tensor products of Banach spaces, projective tensor product of nuclear Fréchet spaces).

Liquid vector spaces. Let $q \in (0, 1]$. The analytic ring of liquid real vector spaces was constructed in [CS20]. The construction of this analytic ring requires a lot of effort due to the non-locally convex functional analysis involved. For instance, if $\mathbb{R}_{<q}$ denotes the analytic ring of $< q$-liquid real numbers, and S is a profinite set, then the free liquid real vector space $\mathbb{R}_{<q}[S]$ is not the naive guess of signed Radon measures on S, but a certain space of $(< q)$-convex Radon measures. The liquid tensor product agrees with the projective tensor product for nuclear Fréchet spaces, as well as for their duals, see [CS22, IV].

Gaseous rings. As we shall see later, one of the main advantages of the new foundations for the theory of analytic rings, based on light condensed sets, is that it is much easier to construct analytic rings out from inverting some concrete maps of modules. The difficulty is then translated in computing the functors of "measures" $A[S]$ for $S \in \mathrm{Prof}^{\mathrm{light}}$. The gaseous ring stack is defined in this way via some universal property in the category of analytic rings. It specializes in both solid and liquid stacks, and its underlying ring $\mathbb{Z}[\widehat{q}]^{\mathrm{gas}} \subset \mathbb{Z}[[q]]$ consists on power series of at most polynomial growth:

$$\mathbb{Z}[\widehat{q}]^{\mathrm{gas}}(*) = \{ \sum_{n >> -\infty} a_n q^n : \exists\, m, k > 0 \text{ such that } \lim_{n \to \infty} |a_n| (n+m)^{-k} = 0 \}.$$

The gaseous ring was motivated from the construction of Tate's elliptic curve $\mathbb{G}_{m,A}^{\mathrm{an}} / q^{\mathbb{Z}}$ in an universal way.

2. Light condensed mathematics

In this talk we will study the basics in light condensed mathematics, this involves light profinite sets, light condensed sets and light condensed abelian groups.

2.1. **Light profinite sets.** Condensed mathematics proposes a better algebraic framework that replaces topological spaces, namely condensed sets. The building blocks of condensed sets are profinite sets that we briefly recall down below.

Proposition 2.1.1. *The following categories are equivalent.*

(1) *The pro-category of finite sets* Pro(Fin) *where maps are given by*

$$\mathrm{Map}(\varprojlim_i S_i, \varprojlim_j T_j) = \varprojlim_j \varinjlim_i \mathrm{Map}(S_i, T_i).$$

(2) *The category of totally disconnected compact Hausdorff spaces with continuous maps.*
(3) *The opposite category of Boolean algebras.*

We let Prof *denote the category of profinite sets, considered as in (1) or (2) above.*

Proof. We just construct the equivalences. From (1) to (2) we take a projective system $\{S_i\}_i$ and pass to the topological space $S = \varprojlim_i S_i$ endowed with the limit topology. From (2) to (1) we take a totally disconnected compact Hausdorff space and consider the projective system $\{S_i\}_{i\in I}$ of finite quotients of S, equivalently, the projective system of partitions of S in clopen subspaces. From (2) to (3) we take a totally disconnected compact Hausdorff space S and consider the Boolean algebra $A = C(S, \mathbb{F}_2)$ of continuous functions from S to $\mathbb{F}_2$. From (3) to (2) we take a Boolean algebra A and consider its spectrum $\operatorname{Spec} A$ as a topological space. □

A delicate issue when working with the category of all profinite sets is that it is not essentially small, i.e. there is not a set of isomorphism classes of objects. On the other hand, all the spaces we actually care about appearing in geometry, topology or analysis (such as manifolds, CW complexes, Banach or Fréchet spaces) admit a norm, and can be recovered within a **set** of smaller profinite sets.

Proposition 2.1.2. *Let S be a profinite set, the following are equivalent.*

(1) *$C(S, \mathbb{Z})$ is countable*
(2) *S is metrizable*
(3) *S is 2-countable*
(4) *S can be written as a sequential limit of finite sets.*

Proof. Urysohn's metrization theorem implies that a compact Hausdorff space is metrizable if and only if it is 2-countable, this shows (2) $\Leftrightarrow$ (3).

(3) $\Leftrightarrow$ (4). By Proposition 2.1.1 the passage from a totally disconnected compact Hausdorff space S to a projective system of finite sets is made by taking the system of partitions of S into clopen subspaces, since S is 2-countable this projective system is countable. Conversely, if $S = \varprojlim_{\mathbb{N}} S_n$ is a sequencial limit of finite sets, taking the fibers of the maps $S \to S_n$ defines a countable basis for the topology of S.

(4) $\Rightarrow$ (1). If $S = \varprojlim_{\mathbb{N}} S_n$, then $C(S, \mathbb{Z}) = \varinjlim_n C(S_n, \mathbb{Z})$ which is countable.

(1) $\Rightarrow$ (3). Finally, if $C(S, \mathbb{Z})$ is countable, then $C(S, \mathbb{F}_2)$ is countable, and $S = \operatorname{Spec} C(S, \mathbb{F}_2)$ has at most countably many clopen subspaces, proving that S is 2-countable. Indeed, clopen subspaces of S are in bijection with the elements of $C(S, \mathbb{F}_2)$. □

Definition 2.1.3. A profinite set is *light* if it satisfies the equivalent conditions of Proposition 2.1.2. We let $\mathrm{Prof}^{\mathrm{light}}$ denote the category of light profinite sets.

Next, we prove some nice features that are special to the category of light profinite sets.

Proposition 2.1.4. *The category of light profinite sets admits countable limits. Moreover, sequential limits of surjections is a surjection.*

Proof. Stability under countable limits follows from Proposition 2.1.2 and that a countable limit of 2-countable topological spaces is 2-countable. Let $S = \varprojlim_n S_n$ be a sequencial limits of surjections, then the map $S \to S_n$ is surjection, namely, given $x_n \in S_n$ take lifts $x_{n+m} \in S_{n+m}$, inductively such that x_{n+m+1} maps to x_m. □

Proposition 2.1.5. *Let S be a light profinite set and let $U \subset S$ be an open subspace. Then U is a countable disjoint union of light profinite sets.*

Proof. Let us write $S = \varprojlim_n S_n$ and let $Z = S \backslash U$. Then $Z = \varprojlim_n Z_n$ with $Z_n \subset S_n$ the image of Z in S_n. Let $\pi_n : S \to S_n$ and $\pi_{m,n} : S_m \to S_n$ denote the projection maps. We define $Y_0 = S_0 \backslash Z_0$ and for $n \geq 1$ we let $Y_n = S_n \backslash (Z_n \cup \pi_{n,n-1}^{-1} Y_{n-1} \cup \cdots \cup \pi_{n,0}^{-1}(Y_0))$. Then

$$U = \bigsqcup_{n\in\mathbb{N}} \pi_n^{-1}(Y_n).$$

□

Proposition 2.1.6. *Let S be a light profinite set. Then S is an injective object in* $\mathrm{Prof}^{\mathrm{light}}$.

Proof. Let $f : X \to Y$ be an injection of light profinite sets and let $g : X \to S$ be a map. The map f is a closed immersion, then we can write it as a sequential limit $\varprojlim_n (f_n : X_n \to Y_n)$ of injective finite sets. We can write the map g as a sequential limit of finite sets $\varprojlim_n (g_n : X_{k_n} \to S_n)$ with k_n some increasing sequence. After taking a subsequence we can assume that $k_n = n$. Then, we can always find a map $h_0 : Y_0 \to S_0$ extending g_0, and provided the extension $h_n : Y_n \to S_n$, we can always find a map $h_{n+1} : Y_{n+1} \to S_{n+1}$ extending g_{n+1} that reduces to h_n in the n-th step. Taking the limit $h = \varprojlim_n h_n$ we get the desired map $h : Y \to S$ extending g. □

Proposition 2.1.7 ([CS19, Theorem 5.4])**.** *Let S be a light profinite set, then the space of continuous functions $C(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module.*

Proof. Let us write $S = \varprojlim_n S_n$ as a sequential limit with surjective maps. We can find compatible sections

$$S_0 \to S_1 \to S_2 \to \cdots$$

and then inductively find compatible sections $S_0 \to S$, $S_1 \to S$, $\cdots$. Then, we know that

$$C(S, \mathbb{Z}) = \varinjlim_n C(S_n, \mathbb{Z}),$$

and we just found compactible sections of $C(S_n, \mathbb{Z}) \to C(S, \mathbb{Z})$, since the modules $C(S_n, \mathbb{Z})$ are free, this shows that $C(S, \mathbb{Z})$ is also free. □

Example 2.1.8. The two examples of light profinite sets that will be the most relevant for us:

(1) The one point compactification of $\mathbb{N}$, namely, $\mathbb{N} \cup \{\infty\}$. It can be written as

$$\mathbb{N} \cup \{\infty\} = \varprojlim_n \{1, 2, \dots, n, \infty\}$$

where for $m \geq n$ the map $\{1, 2, \dots, m, \infty\} \to \{1, 2, \dots, n, \infty\}$ sends all the elements $k \geq n + 1$ to ∞.

(2) The Cantor set $S = \prod_{\mathbb{N}} \{0, 1\}$, it admits a surjective map onto the interval $[0, 1]$ by taking binary decimal expansions.

The relevance of the Cantor set is explained in the following proposition.

Proposition 2.1.9. *A profinite set is light if and only if it admits a surjective map from the Cantor set.*

Proof. Let $S = \varprojlim_n S_n$ be a light profinite set, and let us suppose that $S \to S_n$ is surjective for all n. Then, we can always find a sequence of non-negative integers $(k_n)_{n \in \mathbb{N}}$ and compatible surjection maps for varying n

$$\prod_{m=0}^{k_n} \{0, 1\} \to S_n.$$

Taking the limit we get the desired surjection from the Cantor set. □

2.2. **Light condensed sets.** After the previous preparations of light profinite sets we can finally define light condensed sets (cf. [CS19, Definition 1.2]):

Definition 2.2.1. A light condensed set is a sheaf in the category of light profinite sets for the Grothendieck topology given by finite disjoint unions of jointly surjective maps. More concretely, a condensed set is a functor $T : \mathrm{Prof}^{\mathrm{light,op}} \to \mathrm{Set}$ such that

(1) $T(\emptyset) = *$.
(2) $T(S_1 \sqcup S_2) = T(S_1) \times T(S_2)$.

(3) For all surjective map $S_1 \to S_2$ we have

$$T(S_2) = \mathrm{eq}(T(S_2) \rightrightarrows T(S_2 \times_{S_1} S_2)).$$

We let $\mathrm{CondSet}^{\mathrm{light}}$ denote the category of light condensed sets.

Remark 2.2.2. By Proposition 2.1.4, sequential limits of covers in $\mathrm{Prof}^{\mathrm{light}}$ are covers. In particular, the topos of condensed sets is replete in the sense of [BS14, §3], namely, sequencial limits $T = \varprojlim_n T_n$ of condensed sets with surjective maps are still surjective. Indeed, by definition of the Grothendieck topology, given $S_0 \to T_0$ an S_0-point of T_0 there is a surjective map $S_1 \to S_0$ and a lift $S_1 \to T_1$. Repeating this process we find a compatible sequence of points $S_n \to T_n$ with $S_{n+1} \to S_n$ a surjective map. Then, taking limits $S = \varprojlim_n S_n \to T$, we get a lift of $S_0 \to T_0$ to $S \to T$ and the map $S \to S_0$ is a cover in the Grothendieck topology being surjective by Proposition 2.1.4.

Example 2.2.3. (1) Let T be a light condensed set, then the set $T(\mathbb{N} \sqcup \{\infty\})$ is heuristically the space of convergence sequences with fixed limit, namely, this is exactly the case when T arises from the condensification of a topological space. If $T = \underline{X}$ arises from a Hausdorff space then the set of convergence sequences are determined by its restriction to $\mathbb{N}$, i.e. the map $T(\mathbb{N} \sqcup \{\infty\}) \to T(\mathbb{N})$ is injective. In general, a convergence sequence can have different limits, so the map $T(\mathbb{N} \sqcup \{\infty\}) \to T(\mathbb{N})$ is not necessarily injective.

(2) Let Top denote the category of topological space. We define the condensification functor

$$\underline{(-)} : \mathrm{Top} \to \mathrm{CondSet}^{\mathrm{light}}$$

mapping a topological space X to the condensed set $\underline{X} : S \mapsto C(S, X)$ for $S \in \mathrm{Prof}^{\mathrm{light}}$.

(3) The Yoneda embedding $\mathrm{Prof}^{\mathrm{light}} \to \mathrm{CondSet}^{\mathrm{light}}$ maps a profinite set S to its condensification $\underline{S}$. Since $\mathrm{Prof}^{\mathrm{light}}$ is a small category, any condensed set can be written as a colimit of light profinite sets. More precisely, we have that

$$T = \varinjlim_{S \to T} \underline{S}$$

as a condensed set. From now we will not make further distinction between S and $\underline{S}$ for S a light profinite set.

As we saw in the previous example, there is a natural functor from topological spaces to light condensed sets by mapping from light profinite sets. The following proposition shows that this functor is fully faithful in a reasonable subcategory of topological spaces (cf. [CS19, Proposition 1.7])

Proposition 2.2.4. *The condensification functor has a left adjoint called the "underlying topological space", mapping a condensed set T to the topological space given by*

$$T(*)_{\mathrm{top}} = \varinjlim_{S \to T} S$$

where the colimit is taken in the category of topological spaces. More precisely, $T(*)_{\mathrm{top}}$ has underlying set $T(*)$ and topology determined by the set of maps

$$\bigsqcup_{S \to T} S \to T(*).$$

In particular, the functor $\underline{(-)}$ is fully faithful in metrizably compactly generated spaces (eg. metrizable compact Hausdorff spaces).

Proof. Since $T = \varinjlim_{S \to T} S$ as a condensed set, the statement reduces to the fact that for a profinite set S and a topological space X we have

$$\underline{X}(S) = C(S, X).$$

□

Remark 2.2.5. Let us make more explicit what means to be an epimorphism for topological spaces when considered as condensed sets. Let $X \to Y$ be a map of topological spaces such that their condensification $\underline{X} \to \underline{Y}$ is an epimorphism. This means that for any light profinite set S and any map $f : S \to Y$, there is a surjection from a light profinite set $S' \to S$ and a map $S' \to X$ lifting S. For example, if $X \to Y$ is a surjection of compact Hausdorff spaces then so is its condensification. However, this property does not hold true for example in the case of a inductive limit $\varinjlim_n B_n$ of Banach spaces with injective transitions maps (LB spaces) in the case the maps are not of compact type (the closure of the image of a ball is compact), for the quotient map $\bigsqcup B_n \to \varinjlim B_n$. In other words, the condensification of $\varinjlim_n B_n$ is not necessarily the colimit of the condensification of the Banach spaces B_n unless the maps are compact.

In every topos there is a notion of quasi-compact and quasi-separated objects, in the case of light condensed abelian groups these properties can be stated in more concrete terms.

Definition 2.2.6. A condensed set T is quasi-compact if there is a surjection $S \to T$ from a profinite set. A condenset set T is quasi-separated if for every two maps from profinite sets $S \to T \leftarrow S'$, the fiber product $S \times_T S'$ is quasi-compact.

Remark 2.2.7. By definition, the Grothendieck topology of $\mathrm{Prof}^{\mathrm{light}}$ is finitary, this makes the profinite sets quasi-compact objects in the topos of condensed sets. Moreover, since light profinite sets are stable under countable limits, they are stable under pullbacks and so they are quasi-separated. This makes CondSet a coherent topos. On the other hand, if T is a condensed set and $S, S' \to T$ are maps from profinite sets to T, then $S \times_T S'$ is a subobject of $S \times S$, therefore T is quasi-separated if and only if for all S, S' as before $S \times_T S'$ is also profinite.

We can describe concretely the qcqs objects in CondSet.

Proposition 2.2.8. *Let* $\mathrm{CHaus}^{\mathrm{light}}$ *be the category of metrizable compact Hausdorff spaces. Then the condensification functor induces an equivalence from* $\mathrm{CHaus}^{\mathrm{light}}$ *to the category of qcqs condensed sets. Moreover, the category of quasi-separated condensed sets is equivalent to the ind-category with injective transition maps of metrizable compact Hausdorff spaces* $\mathrm{Ind}_{\mathrm{inj}}(\mathrm{CHaus}^{\mathrm{light}})$.

Proof. First, we claim that a quasi-compact subobject of a light profinite set is necessarily profinite. For this, let $f : S \to S'$ be a map of light profinite sets, we want to see that the image of f is a closed subspace of S'. Let $\mathrm{Im}(f) \subset S'$ be the image as topological space, it is profinite and we know that f factors through the condensification of $\mathrm{Im}(f)$. Then, we are left to show that if f is a surjection of light profinite sets then it is an epimorphism as condensed sets, but this is clear by the definition of the Grothendieck topology of $\mathrm{Prof}^{\mathrm{light}}$.

Let T be a qcqs object in CondSet, then there is a surjection $S \to T$ from a light profinite set such that $S \times_T S$ is also profinite. Then, T arises as the quotient of a light profinite set by a light profinite equivalence relation, making $T(*)_{\mathrm{top}}$ a metrizable compact Hausdorff space, the natural map $\underline{T(*)_{\mathrm{top}}} \to T$ from the adjunction is an equivalence by Remark 2.2.5. Conversely, let X be a metrizable compact Hausdorff space and fix a countable basis $\mathfrak{U}$ of X. Let I denote the countable cofiltered set of finite covers of X by 2 by 2 different elements in $\mathfrak{U}$, and for each $i \in I$ let $S_i = \{U_{j_1}, \dots, U_{j_{k_i}}\}$ be the cover of X. Then $S = \varprojlim_i S_i$ is a light profinite set. We can define $f : S \to X$ by maping a system of open subsets $x = \{\overline{U_{j_i}}\}_{i \in I}$ to its intersection $f(x) = \bigcap_i U_{j_i}$ which is necessarily a point. The map f is then continuous and a surjection from a light profinite set onto X. By Remark 2.2.5 the map of condensed sets $S \to \underline{X}$ is surjective, and the fiber product $S \times_{\underline{X}} S$ is the condensification of the topological fiber product which is a light profinite set, this shows that $\underline{X}$ is qcqs as wanted.

Finally, let T be a quasi-separated light condensed set, and let $S \to T$ be a map from a profinite set S. Then the image X of S in T is qcqs since $S \times_X S = S \times_T S$ is profinite. This shows that T can be written as a union of qcqs condensed sets by injective maps, which produces an object in $\mathrm{Ind}_{\mathrm{inj}}(\mathrm{CHaus}^{\mathrm{light}})$, furthermore, since qcqs condensed sets are compact objects in CondSet this map is fully faithful. Conversely, given a cofiltered diagram $\{X_i\}_i$ of light compact Hausdorff spaces with injective transition maps, the colimit $T = \varinjlim_i \underline{X_i}$ of condensed sets is quasi-separated, namely, given any two maps from profinite sets $S, S' \to T$ there is some i such that S, S' factor through X_i, and $S \times_T S' = S \times_{\underline{X_i}} S'$ is profinite. $\square$

2.3. **Light condensed abelian groups.** Next, we define light condensed abelian groups and prove some of its most important features.

Definition 2.3.1. The category of light condensed abelian groups $\mathrm{CondAb}^{\mathrm{light}}$ is the category of abelian group objects in $\mathrm{CondSet}^{\mathrm{light}}$. Equivalently, it is the category of abelian sheaves on light profinite sets.

Example 2.3.2.
(1) The forgetful functor
$$\mathrm{CondAb}^{\mathrm{light}} \to \mathrm{CondSet}$$
has a left adjoint $T \mapsto \mathbb{Z}[T]$ given by the free abelian group generated by a condensed set. The condensed abelian group $\mathbb{Z}[T]$ is given by the sheafification of the functor mapping a light profinite set S to the free abelian group $\mathbb{Z}[T(S)]$.
(2) Let A be a topological abelian group, then $\underline{A}$ has a natural structure of light condensed abelian group. Indeed, the condensification functor preserves finite limits and the structure of an abelian group for A is encoded in some diagrams such as $+ : A \times A \to A$.
(3) Let $\mathbb{R}$ be the real numbers endowed with the addition and its natural topology, then $\underline{\mathbb{R}}$ is a condensed abelian group. On the other hand, if $\mathbb{R}^{\delta}$ is endowed with the discrete topology then $\underline{\mathbb{R}^{\delta}}$ is another condensed abelian group with same underlying group as $\underline{\mathbb{R}}$. There is an inclusion $\underline{\mathbb{R}^{\delta}} \subset \underline{\mathbb{R}}$ which is not an isomorphism. Indeed, for a light profinite set S we have
$$\underline{\mathbb{R}}/\underline{\mathbb{R}^{\delta}}(S) = C(S, \mathbb{R})/C^{lc}(S, \mathbb{R}),$$
where $C^{lc}(S, \mathbb{R})$ is the space of locally constant functions from S to $\mathbb{R}$.

Theorem 2.3.3. *The category* $\mathrm{CondAb}^{\mathrm{light}}$ *is a Grothendieck abelian category endowed with a natural symmetric monoidal structure and an internal* Hom. *Moreover, it has the following properties*
(1) *Countable products are exact (countable AB4*) and satisfy (AB6).*
(2) *Sequential limits of surjective maps are surjective.*
(3) *The object* $\mathbb{Z}[\mathbb{N} \sqcup \{\infty\}]$ *is internally projective.*

Proof. The fact that $\mathrm{CondAb}^{\mathrm{light}}$ is a Grothendieck abelian category is a general fact about sheaves on abelian groups on a site. It also has a natural tensor product given by the sheafification of the tensor product of presheaves (in particular for $A, B \in \mathrm{CondAb}^{\mathrm{light}}$ we have $(A \otimes B)(*) = A(*) \otimes B(*)$). The internal Hom is just the right adjoint of the tensor product. Point (1) follows from point (2) which is Remark 2.2.2. It is just left to prove point (3).

It suffices to prove that the space of null sequences $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ is internally projective. We want to show that for a surjection $A \to B$ of light condensed abelian groups, and that for all light profinite set S, and a map $g : \mathbb{Z}[S] \otimes N \to B$, there is a dashed arrow making the following diagram commutative
$$\begin{array}{ccc} & & A \\ & \nearrow & \downarrow \\ \mathbb{Z}[S] \otimes P & \xrightarrow{g} & B \end{array}$$
after possibly replacing S by a cover. We have that $\mathbb{Z}[S] \otimes P = \mathbb{Z}[S \times (\mathbb{N} \times \{\infty\})]/(\mathbb{Z}[S \times \{\infty\}])$. Then the map g is the same as a map $S \times (\mathbb{N} \times \{\infty\}) \to B$ sending $S \times \{\infty\}$ to 0. By hypothesis, there is a surjection $f : S' \to S \times (\mathbb{N} \cup \{\infty\})$ and a map $S' \to A$ lifting g. For $n \in \mathbb{N}$ let S'_n be

the fiber over $S \times \{n\}$ (which is still a surjection). By Proposition 2.1.6 we can find retractions $r_n : S' \to S'_n \subset S'$, and construct the following diagram of locally profinite sets

$$\begin{array}{ccc} S' \times \mathbb{N} & \xrightarrow{\bigsqcup_n r_n} & S' \\ & \searrow^{\bigsqcup_n f \circ r_n} & \downarrow f \\ & & S \times (\mathbb{N} \cup \infty). \end{array}$$

We can find a light profinite compactification S'' of $S' \times \mathbb{N}$ such that $S \times \mathbb{N} \to S'$ extends to $S'' \to S'$ (Exercise, construct one of such compactifications). Let D be the boundary of S'', by Proposition 2.1.6 we can find another retraction $r : S'' \to D$. Let $h : S'' \to S' \to A$ be the composite map, then $h - h \circ r$ induces a map

$$\mathbb{Z}[S'']/\mathbb{Z}[D] = \mathbb{Z}[S'] \otimes P \to A$$

that lifts g proving what we wanted. □

Remark 2.3.4. It is surprising that the object $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ is internally projective in $\mathrm{CondAb}^{\mathrm{light}}$. This does not happens at the level of profinite sets, for example the map $(2\mathbb{N} \cup \{\infty\}) \bigsqcup (2\mathbb{N}+1 \cup \{\infty\}) \to \mathbb{N} \cup \{\infty\}$ does not admit a split. This condensed abelian group will be key in the construction of examples on analytic rings.

We can define the condensed cohomology as follows:

Definition 2.3.5. Let $T \in \mathrm{CondSet}^{\mathrm{light}}$ be a light condensed set and M a discrete abelian group, we define the condensed cohomology of T with values in M to be

$$R\Gamma_{\mathrm{cond}}(T, M) := R\mathrm{Hom}(\mathbb{Z}[T], M).$$

Condensed cohomology behaves as expected in good cases.

Proposition 2.3.6 ([CS19, Theorem 3.2])**.** *Let S be a profinite set and M a discrete abelian group, then*

$$R\Gamma_{\mathrm{cond}}(S, M) = C(S, M)$$

is the space of continuous (eq. locally constant) functions from S to M.

Proof. It is clear that $H^0_{\mathrm{cond}}(S, M)$ is just the space of continuous maps from S to M. To show that the higher cohomology groups vanish, it suffices to show that for a cover $S' \to S$ with Čech nerve $(S'^{,\times_S n_1})_{[n] \in \Delta^{\mathrm{op}}}$ the Čech cohomology complex

$$0 \to C(S', M) \to C(S' \times_S S', M) \to \cdots \tag{2.1}$$

is acyclic in cohomological degrees ≥ 1. For this, we can write the surjection $S' \to S$ as a sequential limit of finite sets with surjective maps $\varprojlim_n (S'_n \to S_n)$. Then the Čech complex (2.1) is the colimit of the Čech complexes of the surjections $S'_n \to S_n$, which are acyclic in degrees ≥ 1 since any surjection of finite sets splits. □

Proposition 2.3.7 ([CS19, Theorem 3.2])**.** *Let X be a light compact Hausdorff space and M a discrete abelian group, then there is a natural isomorphism*

$$R\Gamma_{\mathrm{cond}}(X, M) = R\Gamma(X, M)$$

between condensed and Čech cohomology.

Proof. Since X is compact Hausdorff we can formally reduce to the case $M = \mathbb{Z}$. Let $X_{\mathrm{Prof}} := \mathrm{Prof}^{\mathrm{light}}_{/X}$ be the site of light profinite sets over X. Then condensed cohomology of X is the same as the cohomology in X_{Prof}. Let X_{top} be the site consisting on closed subspaces of X with coverings given by finite unions of closed subspaces admitting an open cover refinement. Then Čech cohomology of X is the same as the cohomology on $X_{\top}$. We have a natural morphism of sites

$$\eta : X_{\text{Prof}} \to X_{\text{top}}.$$

It suffices to show that the natural map $\mathbb{Z} \to R\eta_* \mathbb{Z}$ is an isomorphism. This can be proved at stalks, so let $x \in X$, then the stalk $R\eta_*\mathbb{Z}|_x$ is the same as the pushforward of the fiber over x, which is nothing but the condensed cohomology of a point which is $\mathbb{Z}$. □

3. Light solid abelian groups

The theory of solid abelian groups was introduced in [CS19], it plays a fundamental role in non-archimedean analytic geometries and non-archimedean analysis. The category Solid of solid abelian groups is a full subcategory of CondAb, stable under limits, colimits and extensions, and containing $\mathbb{Z}$; it is actually the smallest category satisfying those properties. In its "classical construction" [1] the theory of locally compact abelian groups and its extensions as condensed abelian groups play a key role. However, within the new framework of light condensed mathematics, the theory of solid abelian groups can be formally developed from the more intuitive idea that the "summable sequences" in non-archimedean analysis are precisely the "null-sequences". In the following we will explain how this very simple idea naturally guides us to the correct definition of Solid.

3.1. **Null-sequences and summability.** Let K be a local field and V a Banach space over K. Recall that a *null-sequence* in V is a sequence $(v_n)_{n\in\mathbb{N}}$ converging to 0. Similarly, a *summable sequence* is a sequence $(v_n)_{n\in\mathbb{N}}$ such that the partial sums $\sum_{i=0}^n v_i$ converge to an element in v that we denote by $\sum_n v_n$. One of the first properties that we learn in a course of analysis is that a summable sequence (v_n) has tails $w_n = \sum_{i\geq n} v_n$ converging to 0. In other words, we have a map

$$\{\text{summable sequences}\} \to \{\text{null sequences}\} : (v_n) \mapsto (w_n).$$

On the other hand, given a null sequence $\{w_n\}_{n\in\mathbb{N}}$ we can form the sequence $x_n := w_n - w_{n+1}$ which turns out to be summable in V, namely,

$$v_n := \sum_{i=0}^{n} x_n = w_0 - w_{n+1}$$

and $(v_n)_n$ converges to w_0 as $n \to \infty$. Thus, we get a bijection

$$\{\text{null sequences}\} \to \{\text{summable sequences }\} : (w_n)_n \mapsto (x_n)_n = (w_n - w_{n+1}).$$

Nonetheless, any summable sequence in V is also a null-sequence. The converse does not hold for archimedean fields (eg. $(1/n)_n$), but it does for non-archimedean fields thanks to the ultrametric inequality.

Therefore, a way to isolate non-archimedean analysis from condensed abelian groups is by asking that any null-sequence is summable, namely, that the map

$$1 - S : \{\text{null sequences}\} \to \{\text{null sequences}\},$$

where S is the shift map $(v_n) \mapsto (v_{n+1})$, is a bijection.

In order to formalize this idea, first we need to be able to talk about null-sequences of condensed abelian groups.

Definition 3.1.1. We let $P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$. Given a condensed abelian group A its space of null sequences is given by $\text{Null}(A) = \text{Hom}(P, A)$, we also let $\underline{\text{Null}}(A) := \underline{\text{Hom}}(P, A)$.

[1]If we are allowed to call classical a construction just made around five-six years ago.

Example 3.1.2. We continue in the spirit of Example 2.2.3 (1). For a quasi-separated condensed abelian group A being a null-sequence is an actual property of the underlying sequence, namely, the map

$$\mathrm{Null}(A) \to \mathrm{Map}(\mathbb{N}, A) = \prod_{\mathbb{N}} A(*)$$

is injective. However, for general condensed abelian groups null-sequences are not properties but additional structure you put in the condensed abelian group. As example, let $\mathbb{R}$ be the real numbers with the usual topology, and let $\mathbb{R}^\delta$ be the real numbers with the discrete topology. Then $\mathbb{R}/\mathbb{R}^\delta$, if scary as topological abelian group, is a well defined condensed abelian group, and for any light profinite set S we have that

$$\mathbb{R}/\mathbb{R}^\delta(S) = C(S, \mathbb{R})/C^{lc}(S, \mathbb{R})$$

is the quotient of continuous maps from $S \to \mathbb{R}$ modulo locally constant maps from S to $\mathbb{R}$. Applying this to $S = \mathbb{N} \cup \{\infty\}$ we get that $\mathbb{R}/\mathbb{R}^\delta(S)$ is a non-zero space of null-sequences while $\mathbb{R}/\mathbb{R}^\delta(*) = 0$, this shows that a null-sequence in that non quasi-separated quotient remembers the tails of the virtually zero sequence.

An additional feature for P is that it has a natural structure of algebra making $\mathbb{Z}[T] = \mathbb{Z}[\mathbb{N}] \to \mathbb{P}$ an algebra morphism.

Proposition 3.1.3. *The map addition map*

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

induces an algebra structure on P, we shall denote this algebra by $\mathbb{Z}[\widehat{q}]$.

To prove Proposition 3.1.3, it will suffices to show the following lemma

Lemma 3.1.4. *Consider a surjective map of light profinite sets $S \to S'$ and let $U \subset S'$ be an open subspace such that $S \times_{S'} U \to U$ is an homeomorphism. Let D and D' be the complements of U in S and S' respectively. Then we have a pushout square in* CondSet

$$\begin{array}{ccc} D & \longrightarrow & S \\ \downarrow & & \downarrow \\ D' & \longrightarrow & S'. \end{array}$$

Proof. We have a surjection of condensed sets $S \to S'$ whose Čech fiber is given by $S \times_{S'} S = \Delta S \cup D \times_{D'} D \subset S \times S$. Since $S \to S'$ is surjective, we have that $S' = S/(S \times_{S'} S) = S/(\Delta S \cup D \times_{D'} D)$, which is exactly the pushout $S \bigsqcup_D D'$. ☐

Definition 3.1.5. Let U be a light locally profinite set, i.e. a countable disjoint union of light profinite set. We let $P_U := \mathbb{Z}[U \cup \{\infty\}]/(\infty)$ be the space of "measures on U vanishing at ∞".

Proposition 3.1.6. *Let U be a light locally profinite set, let S be any compactification of U and let D be the boundary, then there is a natural isomorphism $P_U = \mathbb{Z}[S]/\mathbb{Z}[D]$.*

Proof. We have a pushout diagram

$$\begin{array}{ccc} D & \longrightarrow & S \\ \downarrow & & \downarrow \\ * & \longrightarrow & U \cup \{\infty\}, \end{array}$$

applying the left adjoint $\mathbb{Z}[-]$ we get a push out diagram at the level of free modules, which induces the isomorphism

$$\mathbb{Z}[S]/\mathbb{Z}[D] = \mathbb{Z}[S \cup \{\infty\}]/(\infty).$$

☐

Proof of Proposition 3.1.3. We can endow $\mathbb{N}\cup\{\infty\}$ with a structure of additive monoid by declaring $\infty + a = \infty$. Then, $\mathbb{Z}[\mathbb{N}\cup\{\infty\}]$ has a natural algebra structure such that $\mathbb{Z}[\infty]$ is an ideal, this endows P with an algebra structure. More explicitly, consider the addition map

$$(\mathbb{N}\cup\{\infty\})\times(\mathbb{N}\cup\{\infty\})\to\mathbb{N}\cup\{\infty\},$$

it sends the boundary of $\mathbb{N}\times\mathbb{N}$ to the boundary of $\mathbb{N}$, and by Proposition 3.1.6 it defines a map

$$P\otimes P\to P,$$

compatible with the multiplication map $\mathbb{Z}[T]\otimes\mathbb{Z}[T]\to\mathbb{Z}[T]$. It is easy to check that this defines an algebra structure on P. ☐

3.2. **Solid abelian groups form an analytic ring.** Now we define the category of solid abelian groups, for this, note that the solid abelian group P parametrizing null sequences has an endomorphism $\mathrm{Shift}: P\to P$ which is induced from the map of profinite sets $\mathbb{N}\cup\{\infty\}\to\mathbb{N}\cup\{\infty\}$ mapping ∞ to ∞ and n to $n+1$, we call Shift the shift map.

Definition 3.2.1. Consider the map $1-\mathrm{Shift}: P\to P$. A light condensed abelian group A is said *solid* if the natural map

$$\underline{\mathrm{Hom}}(P,A)\xrightarrow{1-\mathrm{Shift}^*}\underline{\mathrm{Hom}}(P,A)$$

is an isomorphism. We let $\mathrm{Solid}\subset\mathrm{CondAb}^{\mathrm{light}}$ denote the full subcategory of (light) solid abelian groups.

More generally, given $C\in\mathscr{D}(\mathrm{CondAb}^{\mathrm{light}})$ an object in the (∞-)derived category of condensed abelian groups, we say that C if solid if the natural map

$$R\underline{\mathrm{Hom}}(P,C)\xrightarrow{1-\mathrm{Shift}^*}R\underline{\mathrm{Hom}}(P,C)$$

is an equivalence. We let $\mathscr{D}(\mathrm{CondAb})^{\square}\subset\mathscr{D}(\mathrm{CondAb}^{\mathrm{light}})$ be the full subcategory of solid objects.

Remark 3.2.2. By Theorem 2.3.3 the object P is internally projective in the category of light condensed abelian groups, in particular there is no difference between the derived or non derived Hom space $\underline{\mathrm{Hom}}(P,A)$. This shows that $\mathrm{Solid}\subset\mathscr{D}(\mathrm{CondAb})^{\square}$.

The main theorem regarding the category of solid abelian groups is the following:

Theorem 3.2.3. *The category* Solid *is a Grothendieck abelian category stable under limits, colimits and extensions in* CondAb. *Furthermore, the following properties hold:*

(1) $\mathbb{Z}\in\mathrm{Solid}$.
(2) *There is a left adjoint* $(-)^{\square}:\mathrm{CondAb}\to\mathrm{Solid}$ *for the inclusion that we call the solidification functor.*
(3) *There is a unique symmetric monoidal structure* $\otimes_{\square}$ *on* Solid *making* $(-)^{\square}$ *symmetric monoidal.*
(4) $\mathbb{R}^{\square}=0$ *(solid abelian groups kill the archimedean theory).*

Moreover, $\mathscr{D}(\mathrm{CondAb})^{\square}$ *is a presentable full subcategory of* $\mathscr{D}(\mathrm{CondAb})$ *stable under limits and colimits, and the following properties are satisfied:*

(5) *The inclusion* $\mathscr{D}(\mathrm{CondAb})^{\square}\to\mathscr{D}(\mathrm{CondAb})$ *has a left adjoint* $(-)^{L\square}$.
(6) *An object* $C\in\mathscr{D}(\mathrm{CondAb})$ *is solid if and only if* $H^i(C)\in\mathrm{Solid}$ *for all* $i\in\mathbb{Z}$, *i.e. the natural t-structure on* $\mathscr{D}(\mathrm{CondAb})$ *induces a t-structure on* $\mathscr{D}(\mathrm{CondAb})^{\square}$.
(7) *For* $C\in\mathscr{D}(\mathrm{CondAb})^{\square}$ *and* $M\in\mathscr{D}(\mathrm{CondAb})$ *we have* $R\underline{\mathrm{Hom}}(M,C)\in\mathscr{D}(\mathrm{CondAb})^{\square}$.
(8) *The category* $\mathscr{D}(\mathrm{CondAb})^{\square}$ *has a unique symmetric monoidal structure* $\otimes^L_{\square}$ *making* $(-)^{L\square}$ *symmetric monoidal.*
(9) *The natural map* $\mathscr{D}(\mathrm{Solid})\to\mathscr{D}(\mathrm{CondAb})$ *of derived categories is fully faithful, and has essential image* $\mathscr{D}(\mathrm{CondAb})^{\square}$.
(10) *The functor* $(-)^{L\square}$ *is the left derived functor of* $(-)^{\square}$.

(11) *The functor* $\otimes^L_{\square}$ *is the left derived functor of* $\otimes_{\square}$.

(12) *For* $S = \varprojlim_n S_n$ *a light profinite set there is a natural equivalence*

$$\mathbb{Z}_{\square}[S] := (\mathbb{Z}[S])^{L\square} \xrightarrow{\sim} \varprojlim_n \mathbb{Z}[S_n] \cong \prod_{\mathbb{N}} \mathbb{Z}.$$

In particular, $\mathbb{Z}_{\square}[S]$ *is a compact projective solid abelian group, and if* S *is infinite* $\mathbb{Z}_{\square}[S]$ *is a compact projective generator of* Solid.

(13) *For* I *and* J *countable sets we have*

$$\prod_I \mathbb{Z} \otimes^L_{\square} \prod_J \mathbb{Z} = \prod_{I\times J} \mathbb{Z}.$$

(14) *The object* $\prod_{\mathbb{N}} \mathbb{Z}$ *is flat in* Solid.

In [CS19] a lot of effort is made in order to prove Theorem 3.2.3 and the only obvious property was point (12), this is because solid abelian groups were constructed by first defining the functor of measures $S \mapsto \mathbb{Z}_{\square}[S]$. Furthermore, property (14) is not true in arbitrary solid abelian groups (counter example due to Effimov). It turns out that with Definition 3.2.1 most of the theorem is immediate.

Proposition 3.2.4. *The category* Solid *is a Grothendieck abelian category. Furthermore, points (1)-(8) hold. Moreover, property (12) implies (9) and (10), and property (13) implies (11).*

Proof. Recall that the category Solid is defined as the full subcategory of condensed abelian groups A such that the map $1 - \mathrm{Shift}^*$ on $\underline{\mathrm{Hom}}(P, A)$ is an isomorphism. Since P is internally projective, this condition is clearly stable under limits, colimits and extensions in CondAb, making Solid an abelian category. The same argument shows that $\mathscr{D}(\mathrm{CondAb})^{\square}$ is stable under limits and colimits in $\mathscr{D}(\mathrm{CondAb})$. It is left to show that Solid and $\mathscr{D}(\mathrm{CondAb})^{\square}$ are presentable, for this, consider $Q = \mathrm{cone}(P \to \varinjlim_{1-\mathrm{Shift}} P)$, then an object C is solid if and only if $R\underline{\mathrm{Hom}}(Q, C) = 0$. Presentability then follows from [Lur09, Theorem 5.5.3.18].

(1) By Proposition 2.3.6 for all $S \in \mathrm{Prof}$ we have that $R\mathrm{Hom}(\mathbb{Z}[S], \mathbb{Z}) = C(S, \mathbb{Z})$ is the space of locally constant functions. This implies that

$$\underline{\mathrm{Hom}}(P, \mathbb{Z}) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}.$$

Then, the action of $1 - \mathrm{Shift}^*$ maps a sequence $(a_0, a_1, \ldots)$ to $(a_0 - a_1, a_1 - a_1, \ldots)$, which clearly has by inverse

$$(b_0, b_1, b_2, \ldots) \mapsto (\sum_{i\geq 0} b_i, \sum_{i\geq 1} b_i, \ldots).$$

since the sequences are eventually zero.

(2) and (5) The existence of the left adjoint follows from the adjoint functor theorem [Lur09, Corollary 5.5.2.9].

(3) and (8) It suffices to show that the kernel of the adjoints $(-)^{\square}$ and $(-)^{L\square}$ are tensor ideals in Solid and $\mathscr{D}(\mathrm{CondAb})^{\square}$ respectively. Let us just explain the proof for $(-)^{L\square}$. Let $A \in \mathscr{D}(\mathrm{CondAb})$ be such that $A^{L\square} = 0$ and let $M \in \mathscr{D}(\mathrm{CondAb})$. To prove that $(M \otimes^L A)^{L\square} = 0$ it suffices to show that for all $B \in \mathscr{D}(\mathrm{CondAb})^{\square}$ we have

$$R\mathrm{Hom}(A \otimes^L M, B) = 0,$$

but we have that

$$R\mathrm{Hom}(A \otimes^L M, B) = R\mathrm{Hom}(A, R\underline{\mathrm{Hom}}(M, B)), \tag{3.1}$$

and $R\underline{\mathrm{Hom}}(M, B)$ is solid by (7), proving that (3.1) vanishes.

(4) Since $\mathbb{R}$ is an algebra and the functor $(-)^{L\square}$ is symmetric monoidal, it suffices to show that
$$\pi_0(\mathbb{R}^{L\square}) = \mathbb{R}^{\square} = 0.$$
Moreover, for this it suffices to show that the unit map $\mathbb{Z} \to \mathbb{R}^{\square}$ is zero. For this, consider the null-sequence in $\mathbb{R}$
$$(1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, \cdots)$$
defining a map $f : P \to \mathbb{R}$. By definition of the solidification, there is an unique map $g : P \to \mathbb{R}^{\square}$ making the following diagram commutative
$$\begin{array}{ccc} P & \xrightarrow{f} & \mathbb{R} \\ \downarrow{\scriptstyle 1-\text{Shift}} & & \downarrow \\ P & \xrightarrow{g} & \mathbb{R}^{\square}. \end{array}$$
Let $[0] : \mathbb{Z} \to P$ be the inclusion in the zero-th component, then $g \circ [0] : \mathbb{Z} \to \mathbb{R}^{\square}$ defines an element x (virtually given by $1 + \frac{1}{2} + \frac{1}{2} + \cdots$). We claim that $x = 2 + x$, this would show that $2 = 0$ and that $\mathbb{R}^{\square} = 0$ since 2 is a unit.

Consider the maps
$$F : \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] : [n] \mapsto [2n + 1] + [2n + 2]$$
$$G : \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] : [n] \mapsto [2n + 1].$$
These maps naturally extend to endomorphisms of P. We claim that we have a commutative diagram
$$\begin{array}{ccc} P & \xrightarrow{F} & P \\ \downarrow{\scriptstyle 1-S} & & \downarrow{\scriptstyle 1-S} \\ P & \xrightarrow{G} & P, \end{array}$$
namely, we have
$$(1-\text{Shift})\circ F([n]) = (1-\text{Shift})([2n+1]+[2n+2]) = [2n+1]-[2n+2]+[2n+2]-[2n+3] = [2n+1]-[2n+3]$$
and
$$G \circ (1 - \text{Shift})([n]) = G([n] - [n + 1]) = [2n + 1] - [2n + 3].$$
On the other hand, we have that $f \circ F = f$, namely it is the sequence
$$((\frac{1}{2} + \frac{1}{2}), (\frac{1}{4} + \frac{1}{4}), (\frac{1}{4} + \frac{1}{4}), (\frac{1}{8} + \frac{1}{8}), (\frac{1}{8} + \frac{1}{8}), (\frac{1}{8} + \frac{1}{8}), (\frac{1}{8} + \frac{1}{8}), \cdots) = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \cdots).$$
By uniqueness of the lift $g : P \to \mathbb{R}^{\square}$, we must have $g \circ G = g$. Then, if g represents the null sequence $(x_0, x_1, x_2, x_3, \cdots)$, we must have $x_n = x_{2n+1}$ for all $n \in \mathbb{Z}$. In particular, $x_0 = x_1$, so that
$$0 = x_0 - x_1 = 1,$$
proving what we wanted.

(6) This follows from the fact that for all $C \in \mathscr{D}(\text{CondAb})$ we have
$$H^i(R\underline{\text{Hom}}(P, C)) = \underline{\text{Hom}}(P, H^i(C)) \text{ for } i \in \mathbb{Z}$$
since P is internally projective.

(7) Let $M \in \mathscr{D}(\text{CondAb})$ and $C \in \mathscr{D}(\text{CondAb})^{\square}$, then the claim follows from the isomorphism
$$R\underline{\text{Hom}}(P, R\underline{\text{Hom}}(M, C)) = R\underline{\text{Hom}}(M, R\underline{\text{Hom}}(P, B)),$$
and the fact that B is solid.

Now let us assume that properties (11) and (12) hold.

(9) The map Solid $\to$ CondAb induces a functor of derived categories $\mathscr{D}(\text{Solid}) \to \mathscr{D}(\text{CondAb})$, by [Lur17, Proposition 1.3.3.7], and since $P^{\square} = \prod_{\mathbb{N}} \mathbb{Z}$ is a compact projective generator of Solid, it suffices to show that for $A \in \text{Solid}$ we have

$$R\underline{\text{Hom}}(P^{\square}, A) = \underline{\text{Hom}}(P^{\square}, A).$$

But we know that $P^{\square} = P^{L\square}$, and by the left adjoints of (2) and (5) we have

$$R\underline{\text{Hom}}(P^{\square}, A) = R\underline{\text{Hom}}(P^{L\square}, A) = R\underline{\text{Hom}}(P, A) = \underline{\text{Hom}}(P, A) = \underline{\text{Hom}}(P^{\square}, A).$$

(10) This follows from the fact that $\mathbb{Z}[S]^{L\square} = \mathbb{Z}[S]^{\square}$ sits in degree zero. Indeed, since both derived categories are right complete, it suffices to show that the restriction of $(-)^{L\square}$ to connective complexes $\mathscr{D}_{\geq 0}(\text{CondAb})$ (i.e. non-negative homological degrees) is the left derived functor. This statement boils to the fact that $(-)^{L\square} : \mathscr{D}_{\geq 0}(\text{CondAb}) \to \mathscr{D}_{\geq 0}(\text{Solid})$ is the left Kan extension of its restriction to the full subcategory of generators $\mathscr{C}^0 = \{\mathbb{Z}[S]\}_{S \in \text{Prof}^{\text{light}}} \subset \mathscr{D}_{\geq 0}(\text{CondAb})$ [2]. In other words, that for $C \in \mathscr{D}_{\geq 0}(\text{CondAb})$ we have

$$C^{L\square} = \varinjlim_{\mathbb{Z}[S] \in \mathscr{C}^0/C} \mathbb{Z}[S]^{L\square} = \varinjlim_{\mathbb{Z}[S] \in \mathscr{C}^0/C} \mathbb{Z}[S]^{\square}.$$

(11) Finally, to show that $\otimes^L_{\square}$ is the left derived functor of $\otimes_{\square}$, it sufices to show that there is a family of compact projective generators $\mathscr{C}^0 \subset \text{Solid}$ stable under the solid tensor product, such that for $A, B \in \mathscr{C}^0$ we have $A \otimes^L_{\square} B = A \otimes_{\square} B$. Taking $\mathscr{C}^0$ as the full subcategory spanned by $\mathbb{Z}_{\square}[S]$ with S light profinite we are done thanks to property (13).

□

Corollary 3.2.5. *Let C be a real condensed vector space. Then $C^{L\square} = 0$.*

Proof. The solidification functor $(-)^{L\square}$ is symmetric monoidal, in particular $\mathbb{R}^{L\square}$ is an algebra and $C^{L\square}$ has a natural $\mathbb{R}^{L\square}$ -module structure. But $\mathbb{R}^{L\square} = 0$, which implies that $C^{L\square} = 0$. □

We have proven most of Theorem 3.2.3, it is left to show points (12)-(14) regarding the explicit description of the free objects $\mathbb{Z}_{\square}[S] := \mathbb{Z}[S]^{L\square}$, their solid tensor products, and the flatness of $\prod_{\mathbb{N}} \mathbb{Z}$ in Solid, we left those properties for the next sections.

3.3. **Computing measures in solid abelian groups.** The objective in this section is to prove the following theorem

Theorem 3.3.1. *Let $S = \varprojlim_n S_n$ be a light profinite set. Then the natural map of solid abelian complexes*

$$\mathbb{Z}[S]^{L\square} \to \varprojlim_n \mathbb{Z}[S_n]$$

is an equivalence. Furthermore, the following hold:

1. $\prod_{\mathbb{N}} \mathbb{Z}$ *is a compact projective generator of* Solid
2. *The natural map* $\mathscr{D}(\text{Solid}) \to \mathscr{D}(\text{CondAb})^{\square}$ *is an equivalence of* ∞*-categories.*
3. *The functor* $(-)^{L\square}$ *is the left derived functor of* $(-)^{\square}$.

By Proposition 3.2.4 it is only left to prove the first assertion of the theorem, this will require some lemmas. Recall that $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ is the solid abelian group parametrizing null-sequences.

First, we see that it suffices to compute the solidification of P in order to compute the solidification of $\mathbb{Z}[S]$ for S a light profinite set.

Lemma 3.3.2. *Let S be a light profinite set, there is a map $P \to \mathbb{Z}[S]$ that induces isomorphisms on solidifications*

$$P^{L\square} \xrightarrow{\sim} \mathbb{Z}[S]^{L\square}.$$

[2] Note that the full subcategory $\mathscr{C}^0 \subset \mathscr{D}_{\geq 0}(\text{CondAb})$ is not a full subcategory of CondAb since the objects of $\mathscr{C}^0$ are not projective

Proof. Let us write $S = \varprojlim_n S_n$ as a limit of finite sets with surjective transition maps and projections $\pi_n : S \to S_n$. We can find a sequence of compatible lifts $S_0 \to S_1 \to S_2 \to \cdots \to S$ with $\iota_n : S_n \to S$. Enumerating $\bigcup_n \iota_n(S_n) \cong \mathbb{N}$ along the previous inclusions, we get an injection $\mathbb{N} \to S$. Then for $a \in \iota_n(S_n) \backslash \iota_{n-1}(S_{n-1}) \subset \mathbb{N}$ consider the element $\iota_n(a) - \iota_{n-1}(a)$. The sequence $(\iota_n(a) - \iota_{n-1}(a))_{n\in\mathbb{N}}$ converges to zero in $\mathbb{Z}[S]$ and defines an injective map $g : P \to \mathbb{Z}[S]$. We claim that g induces an isomorphism after solidification.

We claim that we have a commutative diagram

$$\begin{array}{ccc} P \otimes \mathbb{Z}[S] & \xrightarrow{F} & \mathbb{Z}[S] \\ {\scriptstyle (1-\mathrm{Shift})\otimes \mathrm{id}_S}\uparrow & & \uparrow \\ P \otimes \mathbb{Z}[S] & \xrightarrow{G} & P \end{array} \tag{3.2}$$

where the top horizontal arrow F arises from a map $(\mathbb{N} \times \{\infty\}) \times S \to \mathbb{Z}[S]$ that vanishes at $\infty \times S$. This map is given by the sequence of maps $\{n\} \times S \to \mathbb{Z}[S]$ given by $\mathrm{id}_{\mathbb{Z}[S]}$ if $n = 0$ and $\mathrm{id}_{\mathbb{Z}[S]} - \iota_{n-1} \circ \pi_{n-1}$ if $n \geq 1$, which vanish uniformly on S at ∞. Then, to define the lower horizontal arrow G we need to show that the composite $F \circ (1 - \mathrm{Shift})$ lands in P, but the composite corresponds to the map of condensed sets

$$G : (\mathbb{N} \cup \{\infty\}) \times S \to \mathbb{Z}[S]$$

vanishing at $\infty \times S$, and given by $\iota_{n-1} \circ \pi_{n-1} - \iota_n \circ \pi_n : S \to \mathbb{Z}[S]$ on $\{n\} \times S$ (where we make the convention $\iota_{-1} \circ \pi_{-1} = 0$). In particular, $G(\{n\} \times S)$ lands in P, and so it extends to a map $G : (\mathbb{N} \cup \{\infty\}) \times S \to P$ that vanishes at $\{\infty\} \times S$, producing the desired factorization.

Taking solidifications of (3.2), we get a commutative diagram

$$\begin{array}{ccc} (P \otimes \mathbb{Z}[S])^{L\square} & \xrightarrow{F} & \mathbb{Z}[S]^{L\square} \\ \wr\uparrow & & \uparrow \\ (P \otimes \mathbb{Z}[S])^{L\square} & \xrightarrow{G} & P^{L\square} \end{array} \tag{3.3}$$

where the left vertical arrow is an isomorphism, and the top horizontal arrow has a section induced from the map $\{0\} \times S \to P \otimes \mathbb{Z}[S]$. The previous shows that $\mathbb{Z}[S]^{L\square}$ is a retract of $P^{L\square}$ with idempotent morphism $r : P^{L\square} \to P^{L\square}$. To show that the map is an actual isomorphism we need to show that r is the identity. To prove this last claim, note that the diagram (3.2) restricts to a diagram

$$\begin{array}{ccc} P \otimes P & \xrightarrow{F} & P \\ {\scriptstyle (1-\mathrm{Shift})\otimes \mathrm{id}_P}\uparrow & & {\scriptstyle \mathrm{id}_P}\uparrow \\ P \otimes P & \longrightarrow & P \end{array}$$

via the inclusion $P \subset \mathbb{Z}[S]$. Indeed, the map F is given by the sequence of endomorphisms $\mathrm{id}_{\mathbb{Z}[S]} - \iota_{n-1} \circ \pi_{n-1}$ of $\mathbb{Z}[S]$, which restrict to the endomorphisms $\mathrm{id}_P - \iota_{n-1} \circ \pi_{n-1}$ of P. Taking solidifications we get

$$\begin{array}{ccc} (P \otimes P)^{L\square} & \xrightarrow{F} & P^{L\square} \\ \wr\uparrow & & {\scriptstyle \mathrm{id}_P}\uparrow \\ (P \otimes P)^{L\square} & \longrightarrow & P^{L\square}, \end{array} \tag{3.4}$$

and the idempotent r obtained from (3.3) is the same as the idempotent obtained from (3.4) which is the identity. $\square$

Now, we compute the solidification of P. We apply the same trick as in the proof of Lemma 3.3.2 to replace P by a simpler condensed abelian group.

Lemma 3.3.3. *Let $\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z} = \bigcup_{n\in\mathbb{N}} \prod_{\mathbb{N}} \mathbb{Z} \cap [-n,n] \subset \prod_{\mathbb{N}} \mathbb{Z}$ be the condensed set of bounded sequences of integers. Consider the natural map $P \to \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z}$ induced by the null sequence $e_n \in \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z}$ with $e_n = (0,0,\cdots,0,1,0,\cdots)$, which zero except for a 1 in the n-th component. Then the natural map*

$$P^{L\square} \to (\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z})^{L\square}$$

is an isomorphism.

Proof. We claim that there is a commutative square

$$\begin{array}{ccc} P \otimes \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z} & \xrightarrow{F} & \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z} \\ {\scriptstyle (1-\mathrm{Shift})\otimes \mathrm{id}}\uparrow & & \uparrow \\ P \otimes \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z} & \xrightarrow{G} & P \end{array} \tag{3.5}$$

where the top horizontal arrow F is given by the null-sequence of endomorphisms of $\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z}$ given by the projection $\pi_{\geq n}$ in the $\geq n$-components. To prove the claim, we need to see that the map $G = F \circ (1 - \mathrm{Shift})$ lands in P, but it is given by the null-sequence of endomorphisms of $\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z}$ given by the projections $\pi_n = \pi_{\geq n} - \pi_{\geq n+1}$, whose target is in P. Taking solidifications of (3.5) we get a commutative diagram

$$\begin{array}{ccc} (P \otimes \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z})^{L\square} & \xrightarrow{F} & (\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z})^{L\square} \\ \wr\uparrow & & \uparrow \\ (P \otimes \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z})^{L\square} & \xrightarrow{G} & P^{L\square} \end{array}$$

such that the top horizontal arrow has a section given by the embedding in the 0-th component of the tensor. Then, as in the proof of Lemma 3.3.2, one gets an idempotent endomorphism $r : P^{L\square} \to P^{L\square}$ whose retract is $\mathbb{Z}[S]^{L\square}$, and to see that r is the identity, it suffices to notice that (3.5) restricts to a commutative diagram of the form (3.4), and then one applies the argument as in the proof of Lemma 3.3.2. ☐

Lemma 3.3.4. *The natural map $\prod_{\mathbb{N}}{}^{\mathrm{bnd}}\mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$ induces an isomorphism in solidifications*

$$(\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z})^{L\square} = \prod_{\mathbb{N}} \mathbb{Z}.$$

Proof. Let $\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{R} = \bigsqcup_n \prod_{\mathbb{N}} \mathbb{R} \cap [-n,n]$ be the condensed real vector space. We have isomorphisms of condensed abelian groups

$$\prod_{\mathbb{N}} \mathbb{Z} / \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z} = \prod_{\mathbb{N}} \mathbb{R} / \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{R}.$$

Indeed, this follows from the fact that we have short exact sequences

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{R} \to \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z} \to 0$$

and

$$0 \to \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z} \to \prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{R} \to \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z} \to 0.$$

In particular, the quotient $\prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z}$ can be endowed with an structure of $\mathbb{R}$-condensed vector space, and so its solidification vanishes by Corollary 3.2.5. This shows that

$$(\prod_{\mathbb{N}}^{\mathrm{bnd}} \mathbb{Z})^{L\square} = (\prod_{\mathbb{N}} \mathbb{Z})^{L\square} = \prod_{\mathbb{N}} \mathbb{Z}$$

as wanted. □

Corollary 3.3.5. *Let $S = \varprojlim_n S_n$ be a light profinite set, then we have natural isomorphisms*

$$\mathbb{Z}_{\square}[S] = R\underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$$

and

$$C(S,\mathbb{Z}) = R\underline{\mathrm{Hom}}(\mathbb{Z}_{\square}[S],\mathbb{Z}).$$

Proof. The first isomorphism follows from the fact that $C(S,\mathbb{Z}) = \varinjlim_n C(S_n,\mathbb{Z})$ and that $\mathbb{Z}_{\square}[S] = \varprojlim_n \mathbb{Z}[S_n]$. The second isomorphism follows from the left adjoint $(-)^{L\square}$

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_{\square}[S],\mathbb{Z}) = R\underline{\mathrm{Hom}}(\mathbb{Z}[S],\mathbb{Z}) = C(S,\mathbb{Z}).$$

□

Corollary 3.3.6. *Theorem 3.3.1 holds. Moreover, we have $\prod_{\mathbb{N}} \mathbb{Z} \otimes^L_{\square} \prod_{\mathbb{N}} \mathbb{Z} = \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}$. In particular, $\otimes^L_{\square}$ is the left derived functor of $\otimes_{\square}$.*

Proof. The consequences (1)-(3) of the theorem were proven in Proposition 3.2.4. By Lemmas 3.3.2, 3.3.3 and 3.3.4, we know that $\mathbb{Z}[S]^{\square} \simeq \prod_{\mathbb{N}} \mathbb{Z}$ abstractly as solid abelian groups. Following the explicit isomorphisms constructed in the lemmas, one can verify that the previous isomorphism actually identifies with the natural arrow

$$\mathbb{Z}[S]^{L\square} \xrightarrow{\sim} \varprojlim_n \mathbb{Z}[S_n]. \tag{3.6}$$

More explicitly, this hols true for P by the proof of Lemmas 3.3.3 and 3.3.4. In particular, we have natural isomorphisms $R\underline{\mathrm{Hom}}(\prod_{\mathbb{N}} \mathbb{Z},\mathbb{Z}) = \bigoplus_{\mathbb{N}} \mathbb{Z}$ and $R\underline{\mathrm{Hom}}(\bigoplus_{\mathbb{N}} \mathbb{Z},\mathbb{Z}) = \prod_{\mathbb{N}} \mathbb{Z}$. This shows that the objects $\mathbb{Z}[S]^{L\square}$ are reflexive over $\mathbb{Z}$, and it suffices to show that the map (3.6) becomes an isomorphism after taking duals. This follows from the fact that

$$\begin{aligned} R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{L\square},\mathbb{Z}) &= R\underline{\mathrm{Hom}}(\mathbb{Z}[S],\mathbb{Z}) = C(S,\mathbb{Z}) \\ = \varinjlim_i C(S_i,\mathbb{Z}) &= \varinjlim_i R\underline{\mathrm{Hom}}(\mathbb{Z}[S_i],\mathbb{Z}) = R\underline{\mathrm{Hom}}(\varprojlim_i \mathbb{Z}[S_i],\mathbb{Z}), \end{aligned}$$

where in the last equality we use that $\varprojlim_i \mathbb{Z}[S_i]$ is isomorphic to $\prod_{\mathbb{N}} \mathbb{Z}$ by Proposition 2.1.7.

On the other hand, we have an isomorphism $P \times P \xrightarrow{\sim} P$ given by taking an anti-diagonal enumeration of $\mathbb{N} \times \mathbb{N}$. This shows that

$$\prod_{\mathbb{N}} \mathbb{Z} \otimes^L_{\square} \prod_{\mathbb{N}} \mathbb{Z} \cong (P \otimes P)^{L\square} \cong P^{L\square} \cong \prod_{\mathbb{N}} \mathbb{Z}. \tag{3.7}$$

An explicit description of this enumeration shows that the isomorphism (3.7) is given by the natural map

$$\prod_{\mathbb{N}} \mathbb{Z} \otimes^L_{\square} \prod_{\mathbb{N}} \mathbb{Z} \xrightarrow{\sim} \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}.$$

□

A first interesting property of the solidification functor is that it computes singular cohomology of CW complexes.

Proposition 3.3.7. *Let X be a CW complex, then $\mathbb{Z}[X]^{L\square}$ is equivalent to the complex of singular chains in X.*

Proof. Writing X as a colimit of finite CW complexes it suffices to construct a natural quasi-isomorphism between $\mathbb{Z}[X]^{L\square}$ and the chain complex of X, we can then assume X to be compact. Let $S \to X$ be a surjection from a light profinite set with Čech nerve $S_\bullet \to X$. We have a resolution

$$\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[X] \to 0$$

proving that $\mathbb{Z}[X]^{L\square}$ is given by the connective complex.

$$\cdots \to \mathbb{Z}_\square[S_2] \to \mathbb{Z}_\square[S_1] \to \mathbb{Z}_\square[S_0] \to 0.$$

By Corollary 3.3.5 the complex $\mathbb{Z}[X]^{L\square}$ is reflexive, and to naturally identify it with singular chains it suffices to naturally identify its dual with singular cochains. But

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[X]^{L\square}, \mathbb{Z}) = R\underline{\mathrm{Hom}}(\mathbb{Z}[X], \mathbb{Z}) = R\Gamma_{\mathrm{cond}}(X, \mathbb{Z})$$

is the condensed cohomology of X, that we identified with sheaf cohomology on X by Proposition 2.3.7, and so with singular cochains. □

3.4. **Flatness of $\prod_{\mathbb{N}} \mathbb{Z}$ and the structure of** Solid**.** In this section we prove the last part of Theorem 3.2.3 regarding the flatness of $\prod_{\mathbb{N}} \mathbb{Z}$ as solid abelian group. The proof strategy begins by first describing all the finitely presented solid abelian groups.

Definition 3.4.1. A solid abelian group is said finitely generated if it is a quotient of $\prod_{\mathbb{N}} \mathbb{Z}$. A solid abelian group is said finitely presented if it is a cokernel of a map $\prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$.

Theorem 3.4.2. *The finitely presented objects of* Solid *form an abelian category stable under kernels, cokernels and extensions in* Solid*, such that* $\mathrm{Solid} = \mathrm{Ind}(\mathrm{Solid}^{\mathrm{finpres}})$*. Moreover, any* $M \in \mathrm{Solid}^{\mathrm{finpres}}$ *has a resolution*

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to M \to 0.$$

A first corollary is the flatness of $\prod_{\mathbb{N}} \mathbb{Z}$.

Corollary 3.4.3. *The solid abelian group $\prod_{\mathbb{N}} \mathbb{Z}$ is flat for the solid tensor product.*

Proof. Since $\mathrm{Solid} = \varinjlim(\mathrm{Solid}^{\mathrm{finpres}})$, it suffices to show that for M a finitely presented solid abelian group $M \otimes_\square^L \prod_{\mathbb{N}} \mathbb{Z}$ sits in degree 0. By the Theorem 3.4.2 we have a resolution

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to M \to 0.$$

Tensoring with $\prod_{\mathbb{N}} \mathbb{Z}$, and using Corollary 3.3.6 we see that

$$M \otimes_\square^L \prod_{\mathbb{N}} \mathbb{Z} = \prod_{\mathbb{N}} M$$

which clearly sits in degree 0. □

In order to proof Theorem 3.4.2 we shall need the following lemma.

Lemma 3.4.4. *Any finitely generated submodule of $\prod_{\mathbb{N}} \mathbb{Z}$ is isomorphic to $\prod_I \mathbb{Z}$ with I countable.*

Proof. Let $M \subset \prod_{\mathbb{N}} \mathbb{Z}$ be a finitely generated subobject, then M is the image of a map $f : \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$, which is the dual of a map

$$g : \bigoplus_{\mathbb{N}} \mathbb{Z} \to \bigoplus_{\mathbb{N}} \mathbb{Z}. \tag{3.8}$$

We shall need the following claim:

Claim. Let N be a countable abelian group that embeds in a direct product of $\mathbb{Z}$, then N is free.

Proof of the claim. Let us pick a basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathbb{Q}\otimes N$, and let $N_n = \langle e_0, \ldots, e_n\rangle_{\mathbb{Q}} \cap N$. It suffices to show that each N_n is finite free, namely, we have $N = \varinjlim_n N_n$ and the quotient N_{n+1}/N_n is torsion free. We can assume without loss of generality that $\{e_n\}_{n\in\mathbb{N}} \subset N$. Then, it suffices to prove that $M_n = N_n/\langle e_1, \ldots, e_n\rangle_{\mathbb{Z}}$ is finite. Suppose it is not, then we can find elements $x_m \in M_n$ of exactly b_m torsion for $m \in \mathbb{N}$, so that $b_m \to \infty$ as $m \to \infty$. Taking lifts $y_m \in N_n$ of x_n this implies that $y_m = \sum_{i=0}^n \frac{c_{i,m}}{d_{i,m}} e_i$ with coefficients satisfying the following properties:

- $c_{i,m} = 0$ or $\mathrm{GCD}(c_{i,m} d_{i,m}) = 1$,
- $\mathrm{lcm}(d_{i,m}) = b_m$.

By hypothesis N embeds into $\prod_I \mathbb{Z}$. Then, there is some projection $\prod_I \mathbb{Z} \to \prod_{J\subset I} \mathbb{Z}$ with J finite such that the image of the elements $\{e_1, \ldots, e_n\}$ are linearly independent, proving that for $m >> 0$ the element y_m cannot be mapped into $\prod_{i=0}^k \mathbb{Z}$ as $b_m \to \infty$ as $m \to \infty$, which is a contradiction. This proves the claim. ☐

We can decompose the map $g = j \circ h$ in (3.8) as a split surjection $h : \bigoplus_{\mathbb{Z}} \mathbb{Z} \to M$ and an injection $j : M \to \bigoplus_{\mathbb{N}} \mathbb{Z}$. We can then write short exact sequences

$$0 \to K \to \bigoplus_{\mathbb{N}} \mathbb{Z} \xrightarrow{h} M \to 0$$

and

$$0 \to M \to \bigoplus_{\mathbb{N}} \mathbb{Z} \to Q \to 0$$

with M and K free abelian groups. Taking duals we get exact sequences

$$0 \to M^{\vee} \to \prod_{\mathbb{N}} \mathbb{Z} \to K^{\vee} \to 0$$

and

$$0 \to \underline{\mathrm{Hom}}(Q, \mathbb{Q}) \to \prod_{\mathbb{N}} \mathbb{Z} \to M^{\vee} \to \underline{\mathrm{Ext}}^1(Q, \mathbb{Z}) \to 0.$$

Then, the composite

$$\prod_{\mathbb{N}} \mathbb{Z} \xrightarrow{f} \prod_{\mathbb{N}} \mathbb{Z} \to K^{\vee}$$

is zero and we can assume without loss of generality that $K = 0$ and so g is injective. Thus, we have an exact sequence

$$0 \to \bigoplus_{\mathbb{N}} \mathbb{Z} \xrightarrow{g} \bigoplus_{\mathbb{N}} \mathbb{Z} \to Q \to 0. \tag{3.9}$$

Consider the natural map

$$Q \to \prod_{\mathrm{Hom}(Q,\mathbb{Z})} \mathbb{Z}$$

and let $\overline{Q}$ be its image. By the previous claim $\overline{Q}$ is a free abelian group, and so $Q \to \overline{Q}$ is a split surjection. Thus, by taking out the free direct summand, we can assume without of generality that $\mathrm{Hom}(Q, \mathbb{Z}) = 0$. Then, one actually has that $\underline{\mathrm{Hom}}(Q, \mathbb{Z}) = 0$, namely, the S-valued points of the

$\underline{\mathrm{Hom}}$ space are equal to $\mathrm{Hom}(Q, C(S,\mathbb{Z}))$ and $C(S,\mathbb{Z})$ is a free $\mathbb{Z}$-module. We deduce that the dual of (3.9) is the short exact sequence

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to \underline{\mathrm{Ext}}^1(Q,\mathbb{Z}) \to 0,$$

getting that the image of f is $\prod_{\mathbb{N}} \mathbb{Z}$ as wanted. □

Proof of Theorem 3.4.2. By the proof of Lemma 3.4.4 any finitely presented module $M \in \mathrm{Solid}$ is of the form $M = \prod_I \mathbb{Z} \oplus \underline{\mathrm{Ext}}^1(Q,\mathbb{Z})$ with I a countable set, and Q a countable abelian group such that $\mathrm{Hom}(Q,\mathbb{Z}) = 0$. By taking duals of a free resolution

$$0 \to \bigoplus_{\mathbb{N}} \mathbb{Z} \to \bigoplus_{\mathbb{N}} \mathbb{Z} \to Q \to 0,$$

we get a presentation

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \oplus \prod_{I} \mathbb{Z} \to M \to 0$$

proving the second statement of the theorem. The stability of finitely presented solid modules under kernels, cockernels and extensions is then a standard fact for abelian categories for which finitely presented objects admit a resolution by compact projective generators (i.e. are pseudo-coherent, cf. [Sta22, Tag 064N] for the case of modules over rings). □

3.5. **Examples of solid tensor products.** We finish the discussion of solid abelian groups with some computations of solid tensor products that appear a lot in practice.

Example 3.5.1 (Power series ring)**.** Let $\mathbb{Z}[[q]]$ be the ring of power series in one variable seen as a condensed ring. It is a solid abelian group s since $\mathbb{Z}[[q]] = \varprojlim_n \mathbb{Z}[q]/q^n$ is a limit of discrete modules. Indeed, if $\mathbb{Z}[\widehat{q}] = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ is the algebra of null-sequences, see Proposition 3.1.3, we have $\mathbb{Z}[\widehat{q}]^{L\square} = \mathbb{Z}[[q]]$. Corollary 3.3.6 implies that

$$\mathbb{Z}[[q_1]] \otimes^L_{\square} \mathbb{Z}[[q_2]] = \mathbb{Z}[[q_1, q_2]].$$

On the other hand, the morphism of algebras $\mathbb{Z}[q] \to \mathbb{Z}[[q]]$ is idempotent when seen as solid algebras, namely,

$$\mathbb{Z}[[q]] \otimes^L_{\mathbb{Z}[q]} \mathbb{Z}[[q]] = (\mathbb{Z}[[q_1]] \otimes^L_{\square} \mathbb{Z}[[q_2]]) \otimes^L_{\mathbb{Z}[q_1-q_2]} \mathbb{Z} = \mathbb{Z}[[q_1,q_2]] \otimes^L_{\mathbb{Z}[q_1-q_2]} \mathbb{Z} = \mathbb{Z}[[q_1,q_2]]/^{\mathbb{L}}(q_1-q_2) = \mathbb{Z}[[q]],$$

where $\mathbb{Z}[[q_1,q_2]]/^{\mathbb{L}}(q_1-q_2)$ is the derived quotient, represented by a Koszul complex.

Example 3.5.2 (p-adic integers)**.** The p-adic integers $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n$ is a solid abelian group being a limit of discrete abelian groups. We have a short exact sequence of solid abelian groups

$$0 \to \mathbb{Z}[[X]] \xrightarrow{X-p} \mathbb{Z}[[X]] \to \mathbb{Z}_p \to 0,$$

indeed, this is the limit of the short exact sequences

$$0 \to \mathbb{Z}[X]/X^n \xrightarrow{X-p} \mathbb{Z}[X]/X^n \to \mathbb{Z}/p^n \to 0.$$

Thus, the tensor $\mathbb{Z}_p \otimes^L_{\square} \mathbb{Z}[[Y]]$ is nothing but $\mathbb{Z}_p[[Y]]$.

On the other hand, the tensor product $\mathbb{Z}_p \otimes^L_{\square} \mathbb{Z}_\ell$ is represented by the complex

$$\mathbb{Z}_p[[X]] \xrightarrow{X-\ell} \mathbb{Z}_p[[X]],$$

if $\ell \neq p$ then $\mathbb{Z}_p \otimes^L_{\square} \mathbb{Z}_\ell$ while if $\ell = p$ we get $\mathbb{Z}_p \otimes^L_{\square} \mathbb{Z}_p = \mathbb{Z}_p$. In particular, $\mathbb{Z}_p$ is an idempotent $\mathbb{Z}$-algebra for the solid tensor product. In other words, being a $\mathbb{Z}_p$-module is not additional structure but a property for solid abelian groups!

Example 3.5.3 (I-adically complete modules)**.** Given a discrete ring A and I a finitely generated ideal, there is a notion of being derived I-adically complete (see [Man22, Definition 2.12.3] and [Sta22, Tag 091N]). When $I=(a)$ is generated by a single element, and $A \xrightarrow{a} A$ is the multiplication by a, for an object C in the derived category of (condensed) A-modules being I-adically complete is equivalent to the vanishing of the limit $R\varprojlim_a C = 0$ given by multiplication along the complex $A \xrightarrow{a} A$. If we write $J \to A$ for $A \xrightarrow{a} A$, we can think of J as a generalized Cartier divisor, namely, an invertible A-module together with a map $J \to A$. We can define powers of J by tensoring, obtaining generalized Cartier divisors $J^n \to A$. Then, a A-modules C is derived I-adically complete if the natural nap

$$C \to R\varprojlim C/^{\mathbb{L}} J^n,$$

where the quotient $C/^{\mathbb{L}} J^n$ is the pushout of C along the map of derived rings $A \to A/^{\mathbb{L}} J^n$, where $A/^{\mathbb{L}} J^n$ is the dg-algebra given by the Koszul complex $J \to A$.

By [Man22, Lemma 2.12.9] if A is a finitely generated $\mathbb{Z}$-algebra and N, M are connective derived I-adically complete modules, then $N \otimes^L_{A,\square} M$ is also derived I-adically complete (here the tensor product is the natural one attached for a commutative ring object in Solid, equivalently, it is the solidification of the condensed tensor product over A).

Example 3.5.4 (Tensor product of $\mathbb{Q}_p$-Banach spaces)**.** Specializing Example 3.5.3 to Banach spaces we get the following computation: let I and J be two countable sets, then

$$\widehat{\bigoplus_I} \mathbb{Q}_p \otimes^L_{\mathbb{Q}_p,\square} \widehat{\bigoplus_J} \mathbb{Q}_p = \widehat{\bigoplus_{I\times J}} \mathbb{Q}_p. \tag{3.10}$$

To prove this, since $\widehat{\bigoplus}_I \mathbb{Q}_p = (\widehat{\bigoplus}_I \mathbb{Z}_p)[\frac{1}{p}]$ it suffices to do the analogue computation for $\mathbb{Z}_p$. By Example 3.5.2, the ring $\mathbb{Z}_p$ is an idempotent solid $\mathbb{Z}$-algebra, and so the $\mathbb{Z}$-solid or $\mathbb{Z}_p$-solid tensor products are the same. Then, Example 3.5.3 implies that the solid tensor product

$$\widehat{\bigoplus_I} \mathbb{Z}_p \otimes^L_{\square} \widehat{\bigoplus_J} \mathbb{Z}_p$$

is p-adically complete, and so it is equal to

$$R\varprojlim_n \Big(\bigoplus_I \mathbb{Z}/p^n \otimes^L_{\square} \bigoplus_J \mathbb{Z}/p^n\Big) = R\varprojlim_n \bigoplus_{I\times J} \mathbb{Z}/p^n = \widehat{\bigoplus_{I\times J}} \mathbb{Z}_p.$$

For a more direct proof of this fact see [RJRC22, Lemma 3.13].

Example 3.5.5 (Tensor product Fréchet spaces)**.** A Fréchet $\mathbb{Q}_p$-vector space is by definition a sequential limit $F = \varprojlim_n V_n$ of Banach spaces, in particular they are naturally solid $\mathbb{Q}_p$-vector spaces. If $G = \varprojlim_n W_n$ is another Fréchet space then

$$F \otimes^L_{\square} G = \varprojlim_n (V_n \otimes_{\square} W_n)$$

is the projective tensor product in classical functional analysis. In particular, we have that for I and J countable sets we get

$$\prod_I \mathbb{Q}_p \otimes^L_{\square} \prod_J \mathbb{Q}_p = \prod_{I\times J} \mathbb{Q}_p.$$

For a proof of this fact see for example [RJRC22, Lemma 3.28].

4. Analytic rings

The building blocks of algebraic geometry are given by commutative rings. In analytic geometry the building blocks are the so called "analytic rings". The notion of analytic ring arises from the following desiderata:

- An analytic ring A should have an underlying "topological" or condensed ring $A^{\triangleright}$.
- An analytic rings A should be endowed with a category of complete A-modules Mod_A, and with a complete tensor product $\otimes_A$.

In the next section we introduce analytic rings and prove some of their most fundamental properties. We will see how the new light foundations of the theory help to construct new examples of analytic rings.

4.1. First definitions and properties.

We want to define building blocks for analytic geometry for which we can naturally attach a category of "complete modules". It turns out that in condensed mathematics a category of complete modules for a condensed ring is additional datum; given a condensed ring A there could be very different ways to complete condensed A-modules, and none of them should have a preference. Nonetheless, once a category of "complete modules" is fixed, being a complete module should be just a property.

On the other hand, derived algebraic geometry [Lur04, Toe14] has shown that the correct framework to study geometric properties of algebraic varieties such as intersections is within higher category theory. In analytic geometry the requirement of higher category theory and higher algebra (taken in the form of [Lur09, Lur17, Lur18]) is even more notorious: even open localizations of rigid or complex spaces are not going to be flat. In particular, the only way to obtain actually useful new descent results is by looking at the ∞-derived categories of modules.

This desiderata for the notion of analytic ring is formalized in the following definition (see [CS20, Definition 12.1 and Proposition 12.20] and [Man22, Definition 2.3.1]).

Definition 4.1.1 (Analytic ring)**.** An uncompleted analytic ring is a pair $A = (A^{\triangleright}, \mathscr{D}(A))$ consisting on a condensed animated ring $A^{\triangleright}$ and a full subcategory $\mathscr{D}(A) \subset \mathscr{D}(A^{\triangleright})$ of the ∞-category of condensed $A^{\triangleright}$-modules satisfying the following properties.

1. $\mathscr{D}(A)$ is stable under limits and colimits in $\mathscr{D}(A^{\triangleright})$ and there is a left adjoint $F : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}(A)$ for the inclusion.
2. $\mathscr{D}(A)$ is linear over $\mathscr{D}(\mathrm{CondAb})$ [3]. More precisely for all $C \in \mathscr{D}(\mathrm{CondAb})$ and $M \in \mathscr{D}(A)$ the object $R\underline{\mathrm{Hom}}_{\mathbb{Z}}(C, M)$ is in $\mathscr{D}(A)$.
3. The left adjoint F sends connective objects to connective objects. In particular, $\mathscr{D}(A)$ has a natural t-structure induced from $\mathscr{D}(A^{\triangleright})$ (see Proposition 4.1.7).

- We say that A is an analytic ring structure of $A^{\triangleright}$. Finally, we say that A is an analytic ring if in addition $A^{\triangleright} \in \mathscr{D}(A)$. We often write $A \otimes_{A^{\triangleright}} -$ for the left adjoint F (note the drop of derived notation).
- Given T a condensed (animated) set we let $A[T] := A \otimes_{A^{\triangleright}} A^{\triangleright}[T]$, where $A^{\triangleright}[T]$ is the free $A^{\triangleright}$-module generated by T.
- A morphism of analytic rings $f : A \to B$ is a morphism of animated condensed rings $f : A^{\triangleright} \to B^{\triangleright}$ such that the forgetful functor $f_* : \mathscr{D}(B^{\triangleright}) \to \mathscr{D}(A^{\triangleright})$ sends $\mathscr{D}(B)$ to $\mathscr{D}(A)$.
- We let AnRing^{un} denote the ∞-category of uncompleted analytic rings. Let $\mathrm{AnRing} \subset \mathrm{AnRing}^{un}$ be the full subcategory of (completed) analytic rings.

Remark 4.1.2. Condition (2) of Definition 4.1.1 is equivalent to the following:

(2') For all $C \in \mathscr{D}(A^{\triangleright})$ and $M \in \mathscr{D}(M)$ then $R\underline{\mathrm{Hom}}_{A^{\triangleright}}(C, M)$ is in $\mathscr{D}(A)$.

[3] This condition implies that $\mathscr{D}(A)$ is actually enriched in condensed abelian groups. It can be heuristically thought as a suitable "continuity" or "condensed" condition for $\mathscr{D}(A)$.

Indeed, it suffices to check the condition (2') and (2) on generators of $\mathscr{D}(A^{\triangleright})$ and $\mathscr{D}(\mathrm{CondAb})$ respectively. Then we can suppose without loss of generality that $C = A^{\triangleright}[S]$ or $C = \mathbb{Z}[S]$ for $S \in \mathrm{Prof}^{\mathrm{light}}$. In this case we have

$$R\underline{\mathrm{Hom}}_{A^{\triangleright}}(A^{\triangleright}[S], M) = R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], M).$$

Remark 4.1.3. Recall that in the new foundations we work with light profinite sets, and so for a condensed animated ring $A^{\triangleright}$ the category $\mathscr{D}(A^{\triangleright})$ is presentable. In particular, condition (1) of Defintion 4.1.1 implies that the category $\mathscr{D}(A)$ is an accesible localization of $\mathscr{D}(A^{\triangleright})$, and so presentable by [Lur09, Proposition 5.5.4.15] (the small class of morphisms we invert can be taken as the maps $A^{\triangleright}[S] \to A[S]$ for $S \in \mathrm{Prof}^{\mathrm{light}}$).

Example 4.1.4. So far we have seen essentially only two examples of analytic rings.

1. (1) The initial analytic ring is $\mathbb{Z} = (\mathbb{Z}, \mathscr{D}(\mathrm{CondAb}))$, the ring of condensed integers. More generally, given B a condensed animated ring, we let $B = (B, \mathscr{D}(B))$ denote the trivial analytic ring structure on B.
2. (2) A more "complete" analytic ring is $\mathbb{Z}_{\square} = (\mathbb{Z}, \mathscr{D}(\mathrm{Solid}))$, the ring of solid integers. Later in §5 we shall introduce more examples of analytic rings arising in solid geometry.
3. (3) Other analytic rings are the liquid rings of [CS20] and the gaseous ring of Example 1.4; these rings are global in the sense that they define analytic ring structures over the subring $\mathbb{Z}[\widehat{q}] \subset \mathbb{Z}[[q]]$ of null-sequences that specializes to analytic ring structures over all type of local fields (reals, p-adics, and modulo p).
4. (4) In Section 4.6 we discuss a general way to construct analytic rings. This addresses a problem in the previous foundations of condensed mathematics, namely, the difficulty of constructing analytic rings.

Condensed rings embed fully faithful into analytic rings via the trivial analytic ring structure.

Proposition 4.1.5. *The functor $F : \mathrm{Cond\,Ani\,Ring} \to \mathrm{AnRing}^{un}$ mapping an animated condensed ring $A^{\triangleright}$ to $(A, \mathscr{D}(A^{\triangleright}))$ is fully faithful. Moreover, F has a right adjoint mapping an uncompleted analytic ring B to its underlying condensed ring $B^{\triangleright}$.*

Proof. By definition, given two uncompleted analytic rings A and B the mapping space $\mathrm{Map}_{\mathrm{AnRing}^{un}}(A, B)$ is the full subspace of $\mathrm{Map}_{\mathrm{CondRing}}(A^{\triangleright}, B^{\triangleright})$ such that the forgetful functor $\mathscr{D}(B^{\triangleright}) \to \mathscr{D}(A^{\triangleright})$ sends complete objects to complete objects. If A has the trivial analytic ring structure this condition is tautological, proving that

$$\mathrm{Map}_{\mathrm{AnRing}^{un}}(A^{\triangleright}, B) = \mathrm{Map}_{\mathrm{CondRing}}(A^{\triangleright}, B^{\triangleright})$$

proving the fully-faithfulness and the adjunction. □

The category of complete modules of an uncompleted analytic ring has a natural symmetric monoidal structure.

Proposition 4.1.6 ([CS20, Proposition 12.4] and [Man22, Proposition 2.3.2])**.** *The category $\mathscr{D}(A)$ has a unique symmetric monoidal structure $\otimes_A$ making $A \otimes_{A^{\triangleright}} - : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}(A)$ symmetric monoidal. Moreover, given $A \to B$ a morphism of analytic rings, the functor*

$$\mathscr{D}(A^{\triangleright}) \xrightarrow{B^{\triangleright}\otimes_{A^{\triangleright}} -} \mathscr{D}(B^{\triangleright}) \xrightarrow{B\otimes_{B^{\triangleright}}} \mathscr{D}(B)$$

factors (uniquely) through a functor

$$\mathscr{D}(A^{\triangleright}) \xrightarrow{A\otimes_{A^{\triangleright}} -} \mathscr{D}(A) \xrightarrow{B\otimes_{A} -} \mathscr{D}(B).$$

The functor $B \otimes_A -$ is the left adjoint of the forgetful functor $\mathscr{D}(B) \to \mathscr{D}(A)$.

Proof. To show that $\mathscr{D}(A)$ has a natural symmetric monoidal structure such that $A\otimes_{A^\triangleright}$ is symmetric monoidal, it suffices to show that the kernel K of the completion functor is a $\otimes$-ideal by [NS18, Theorem I.3.6]. Let $M \in \mathscr{D}(A^\triangleright)$ be such that $A \otimes_{A^\triangleright} M = 0$ and let $C \in \mathscr{D}(A^\triangleright)$. Then, for $N \in \mathscr{D}(A)$, we have

$$\begin{aligned} R\underline{\mathrm{Hom}}_{A^\triangleright}(A \otimes_{A^\triangleright} (C \otimes_{A^\triangleright} M), N) &= R\underline{\mathrm{Hom}}_{A^\triangleright}(C \otimes_{A^\triangleright} M, N) \\ &= R\underline{\mathrm{Hom}}_{A^\triangleright}(M, R\underline{\mathrm{Hom}}_{A^\triangleright}(C, N)) \\ &= R\underline{\mathrm{Hom}}_{A^\triangleright}(A \otimes_{A^\triangleright} M, R\underline{\mathrm{Hom}}_{A^\triangleright}(C, N)) \\ &= 0, \end{aligned}$$

where the first two equalities are the obvious adjunctions, and the third equality follows since $R\underline{\mathrm{Hom}}_{A^\triangleright}(C, N)$ is A-complete by (2) of Definition 4.1.1 (cf. Remark 4.1.2). The previous shows that $A \otimes_{A^\triangleright} (C \otimes_{A^\triangleright} M) = 0$ as wanted.

Now, in order to see that the composite

$$\mathscr{D}(A^\triangleright) \xrightarrow{B^\triangleright \otimes_{A^\triangleright} -} \mathscr{D}(B^\triangleright) \xrightarrow{B \otimes_{B^\triangleright}} \mathscr{D}(B)$$

factors through $\mathscr{D}(A)$, it suffices to see that it kills the kernel of $A\otimes_{A^\triangleright}$ (then it would be immediate that the resulting functor is symmetric monoidal). Let $M \in \mathscr{D}(A)$ be an object killed by A-completion and let $K \in \mathscr{D}(B)$, then

$$\begin{aligned} R\underline{\mathrm{Hom}}_{B^\triangleright}(B \otimes_{B^\triangleright} (B^\triangleright \otimes_{A^\triangleright} M), K) &= R\underline{\mathrm{Hom}}_{B^\triangleright}(B^\triangleright \otimes_{A^\triangleright} M, K) \\ &= R\underline{\mathrm{Hom}}_{A^\triangleright}(M, K) \\ &= R\underline{\mathrm{Hom}}_{A^\triangleright}(A \otimes_{A^\triangleright} M, K) \\ &= 0, \end{aligned}$$

where the first three equalities are adjunctions, and the last follows since K is an A-complete module by definition of analytic ring. ☐

Completion of modules for analytic rings can be detected at the level of cohomology groups.

Proposition 4.1.7 ([CS20, Proposition 12.4])**.** *Let A be an analytic ring. An object $M \in \mathscr{D}(A^\triangleright)$ is A-complete if and only if $\pi_i(M) = H^{-i}(M)$ is A-complete for all $i \in \mathbb{Z}$.*

Proof. Let us first show the statement for connective objects (i.e. concentrated in positive homological degrees). Let $M \in \mathscr{D}(A)_{\geq 0}$ and consider the fiber sequence

$$\pi_{\geq 1} M \to M \to \pi_0 M.$$

Taking completions we get a fiber sequence

$$A \otimes_{A^\triangleright} (\pi_{\geq 1} M) \to M \to A \otimes_{A^\triangleright} (\pi_0 M).$$

Since completion preserves connective objects, taking ≥ 1-truncations we get a map

$$A \otimes_{A^\triangleright} (\pi_{\geq 1} M) \to \pi_{\geq 1} M$$

which exhibits $\pi_{\geq 1} M$ as a retract of $A \otimes_{A^\triangleright} (\pi_{\geq 1} M)$. Since $\mathscr{D}(A)$ is stable under colimits we deduce that $\pi_{\geq 1} M$ and so $\pi_0(M)$ are in $\mathscr{D}(A)$. An inductive argument shows that $\pi_i(M)$ is A-complete for all $i \geq 0$. Conversely, let $M \in \mathscr{D}_{\geq 0}(A^\triangleright)$ be such that all its homotopy groups $\pi_i M$ are A-complete. Then $M = \varprojlim_n \tau_{\leq n} M$ is the limit of its Postnikov tower. By induction, each truncation $\tau_{\leq n} M$ is A-complete and then so is M since $\mathscr{D}(A)$ is stable under limits.

We now prove the general case. Let $M \in \mathscr{D}(A)$, then we can write

$$M = \varinjlim_n \tau_{\geq -n} M,$$

and by the connective case it suffices to show that each $\tau_{\geq -n} M$ is A-complete. Since A-completion preserves connective objects, $\tau_{\geq -n} M$ is a retract of $A \otimes_{A^\triangleright} (\tau_{\geq -n} M)$, and so A-complete since $\mathscr{D}(A)$

is stable under colimits. Conversely, suppose that $M \in \mathscr{D}(A^{\triangleright})$ is such that $\pi_i(M)$ is A-complete for all $i \in \mathbb{Z}$. By the connective case we know that $\tau_{\geq -n}M$ is A-complete for all $n \in \mathbb{N}$. The proposition follows by writing $M = \varinjlim_n \tau_{\geq -n}M$. □

Our next goal is to show that analytic rings admit small colimits. As a first approximation let us show that uncompleted analytic rings have small colimits. First we will recall induced analytic structures [Man22, Definition 2.3.13].

Lemma 4.1.8 (Induced analytic structure)**.** *Let A be an uncomplete analytic ring and let B be an animated $A^{\triangleright}$-algebra. Then there is a natural induced analytic structure $B_{A/}$ on B such that $\mathscr{D}(B_{A/}) \subset \mathscr{D}(B)$ is the full subcategory of B-modules whose underlying $A^{\triangleright}$-module is A-complete. The uncompleted analytic ring $B_{A/}$ is the pushout $A \otimes_{A^{\triangleright}} B$, where $A^{\triangleright}$ and B are endowed with the trivial analytic ring structure.*

Proof. We want to see that $B_{A/}$ defines an (uncompleted) analytic ring structure on B. Stability under limits and colimits is clear since the forgetful functor $\mathscr{D}(B) \to \mathscr{D}(A^{\triangleright})$ commutes with limits and colimits. On the other hand, the inclusion $\mathscr{D}(B_{A/}) \to \mathscr{D}(B)$ has by left adjoint

$$B_{A/} \otimes_B - = A \otimes_{A^{\triangleright}} -$$

which still sends B-modules to B-modules as $A \otimes_{A^{\triangleright}} -$ is symmetric monoidal. Indeed, let $C \in \mathscr{D}(B^{\triangleright})$ and $K \in \mathscr{D}(B)$. We have a natural equivalence of $B^{\triangleright}$-modules thanks to the Barr construction

$$C = B^{\triangleright} \otimes_{B^{\triangleright}} C = \varinjlim_{[n] \in \Delta^{\mathrm{op}}} B^{\triangleright, \otimes_{A^{\triangleright}} n+1} \otimes_{A^{\triangleright}} C.$$

Therefore,

$$\begin{aligned} R\underline{\mathrm{Hom}}_B(C, K) &= R\underline{\mathrm{Hom}}_B(\varinjlim_{[n]\in\Delta^{\mathrm{op}}} B^{\otimes_{A^{\triangleright}} n+1} \otimes_{A^{\triangleright}} C, K) \\ &= \varprojlim_{[n]\in\Delta} R\underline{\mathrm{Hom}}_B(B^{\otimes_{A^{\triangleright}} n+1} \otimes_{A^{\triangleright}} C, K) \\ &= \varprojlim_{[n]\in\Delta} R\underline{\mathrm{Hom}}_{A^{\triangleright}}(B^{\otimes_{A^{\triangleright}} n} \otimes_{A^{\triangleright}} C, K) \\ &= \varprojlim_{[n]\in\Delta} R\underline{\mathrm{Hom}}_{A^{\triangleright}}((A \otimes_{A^{\triangleright}} B)^{\otimes_A n} \otimes_A (A \otimes_{A^{\triangleright}} C), K) \end{aligned}$$

where in the first equivalence we use the Barr construction of the tensor product, the second equivalence follows since $R\underline{\mathrm{Hom}}$ commutes with limits, the third follows by $\otimes$-adjuction, the fourth follows from adjuction of A-completion and the fact that K is A-complete. On the other hand, the same computation shows that

$$\begin{aligned} R\underline{\mathrm{Hom}}_B(A \otimes_{A^{\triangleright}} C, K) &= \varprojlim_{[n]\in\Delta} R\underline{\mathrm{Hom}}_{A^{\triangleright}}((A \otimes_{A^{\triangleright}} B)^{\otimes_A n} \otimes_A (A \otimes_{A^{\triangleright}} (A \otimes_{A^{\triangleright}} C)), K) \\ &= \varprojlim_{[n]\in\Delta} R\underline{\mathrm{Hom}}_{A^{\triangleright}}(A \otimes_{A^{\triangleright}} (B^{\otimes_{A^{\triangleright}} n} \otimes_{A^{\triangleright}} C), K) \\ &= R\underline{\mathrm{Hom}}_B(C, K), \end{aligned}$$

where the second equality follows since A-completion is symmetric monoidal and idempotent. This proves that $B_{A/} \otimes_B C = A \otimes_{A^\triangleright} C$ as wanted.

Stability under $R\underline{\mathrm{Hom}}(C, -)$ for $C \in \mathscr{D}(\mathrm{CondAb})$ is obvious. It is also clear that the left adjoint $B_{A/} \otimes_B -$ sends connective objects to connective objects. Thus we have proven that $B_{A/}$ is an analytic ring.

Let us now check that $B_{A/} = A \otimes_{A^\triangleright} B$ as uncompleted analytic rings. Let C be an uncomplete analytic ring. Since B and $A^\triangleright$ have the trivial analytic ring structure, Proposition 4.1.5 implies that a map $B \to C$ is just given by a map of condensed rings $B \to C^\triangleright$. Thus, it suffices to see that the following diagram of mapping spaces is cartesian

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{AnRing}^{uc}}(B_{A/}, C) & \longrightarrow & \mathrm{Map}_{\mathrm{CondRing}}(B, C^\triangleright) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{AnRing}^{uc}}(A, C) & \longrightarrow & \mathrm{Map}_{\mathrm{CondRing}}(A^\triangleright, C^\triangleright). \end{array} \tag{4.1}$$

The bottom horizontal map of (4.1) is an inclusion. Then the pullback $\mathscr{C}$ of (4.1) is the full subspace of $\mathrm{Map}_{\mathrm{CondRing}}(B, C^\triangleright)$ consisting on those maps $B \to C^\triangleright$ of $A^\triangleright$-algebras such that the forgetful functor $\mathscr{D}(C^\triangleright) \to \mathscr{D}(B^\triangleright)$ sends C-complete objects to A-complete modules. But this is by definition the mapping space $\mathrm{Map}_{\mathrm{AnRing}^{un}}(B_{A/}, C)$, proving what we wanted. □

A second important kind of colimit of uncompleted analytic rings is obtained by taking intersections of analytic ring structures.

Lemma 4.1.9. *Let $A^\triangleright$ be a condensed animated ring and let $\{A_i\}_{i\in I}$ be a diagram of (uncompleted) analytic ring structures over $A^\triangleright$. Then the pair $B = (A^\triangleright, \bigcap_i \mathscr{D}(A_i))$ is an (uncompleted) analytic ring representing the colimit $\varinjlim_i A_i$ in the category $\mathrm{AnRing}^{(un)}_{A^\triangleright/}$ of (uncompleted) analytic rings over $A^\triangleright$.*

Proof. Let B denote the pair $(A^\triangleright, \bigcap_i \mathscr{D}(A_i))$ where the intersection takes place in $\mathscr{D}(A^\triangleright)$. Note that conditions (1)-(3) of Definition 4.1.1 are stable under intersection; conditions (2) and (3) are obvious once (1) is proven. Stability under limits and colimits in (1) is clear. The existence of the left adjoint in (1) follows from the adjoint functor theorem [Lur09, Corollary 5.5.2.9]. Indeed, since all the functors involved in the diagram I are accessible localizations of $\mathscr{D}(A^\triangleright)$, all the categories $\mathscr{D}(A_i)$ are presentable by Remark 4.1.3, and then so is its intersection by [Lur09, Theorem 5.5.3.18]. Moreover, if $A^\triangleright$ is A_i-complete for all i, it is also B-complete proving that B is an analytic ring if all the A_i are so.

It is left to show that B is the colimit of the diagram A_i in the category of (uncompleted) analytic rings over $A^\triangleright$. This follows from the fact that for any $C \in \mathrm{AnRing}^{un}$ the maps

$$\mathrm{Map}_{\mathrm{AnRing}^{un}}(A_i, C) \to \mathrm{Map}_{\mathrm{CondRing}}(A^\triangleright, C^\triangleright)$$

are fully-faithful embeddings for all i, and then so its its limit. Then, the limit $\varprojlim_i \mathrm{Map}_{\mathrm{AnRing}^{un}}(A_i, C)$ over $\mathrm{Map}_{\mathrm{CondRing}}(A^\triangleright, C^\triangleright)$ is the full-subanima of $\mathrm{Map}_{\mathrm{CondRing}}(A^\triangleright, C^\triangleright)$ whose connected components are those maps $A^\triangleright \to C^\triangleright$ such that the forgetful functor sends C-complete modules to A_i-complete modules for all i. This is exactly the mapping space $\mathrm{Map}_{\mathrm{AnRing}^{un}}(B, C)$ proving what we wanted. □

We can finally prove the existence of colimits in uncomplete analytic rings.

Proposition 4.1.10. *The category AnRing^{un} of uncompleted analytic rings have small colimits. More precisely, let $\{A_i\}_I$ be a diagram of uncompleted analytic rings. Then $B = \varinjlim_i A_i$ is the uncompleted analytic ring with underlying ring $B^\triangleright = \varinjlim_i A_i^\triangleright$ and with category of complete modules $\mathscr{D}(B) \subset \mathscr{D}(B^\triangleright)$ given by those $B^\triangleright$-modules M whose restrictions to an $A_i^\triangleright$-module is A_i-complete for all i.*

Proof. First, let us show that the pair $B = (B^{\triangleright}, \mathscr{D}(B))$ constructed in the statement of the proposition is an analytic ring. This follows from the fact that B can be written as the colimit

$$B = \varinjlim_i B^{\triangleright}_{A_i/},$$

of uncompleted analytic ring structures over $B^{\triangleright} = \varinjlim_i A_i^{\triangleright}$ (Proposition 4.1.9), where $B^{\triangleright}_{A_i/}$ is the induced analytic ring structure of Lemma 4.1.8.

Let us now consider the underlying diagram of condensed animated rings $\{A_i^{\triangleright}\}_i$. Let $C \in \mathrm{AnRing}^{un}$. By definition of the category of analytic rings the limit

$$\varprojlim_i \mathrm{Map}_{\mathrm{AnRing}^{un}}(A_i, C) \tag{4.2}$$

is a full-subanima of the space

$$\varprojlim_i \mathrm{Map}_{\mathrm{AnRing}^{un}}(A_i^{\triangleright}, C^{\triangleright}) = \mathrm{Map}(B^{\triangleright}, C^{\triangleright}).$$

Furthermore, it is the full subanima of connected components consisting on those maps $B^{\triangleright} \to C^{\triangleright}$ for which a complete C-module is A_i-complete, equivalenty, for which a complete C-module is $B^{\triangleright}_{A_i/}$-complete. This shows that (4.2) is the full anima $\mathrm{Map}_{\mathrm{AnRing}^{un}}(B, C) \subset \mathrm{Map}(B^{\triangleright}, C^{\triangleright})$, proving that $B = \varinjlim_i A_i$ as wanted. $\square$

A first consequence of the previous lemma is the stability of analytic rings under sifted colimits in the category of uncompleted analytic rings.

Corollary 4.1.11. *The ∞-category* AnRing *of analytic rings is stable under sifted colimits in* AnRing^{un}*. Moreover, let $B = \varinjlim_i A_i$ be a sifted colimit of uncompleted analytic rings. Then for $S \in \mathrm{Prof}^{\mathrm{light}}$ we have*

$$B[S] = \varinjlim_i A_i[S]$$

Proof. It suffices to show the second claim, namely, if the terms in the sifted colimits are analytic rings we have

$$B[*] = \varinjlim_i A_i[*] = \varinjlim_i A_i^{\triangleright} = B^{\triangleright},$$

proving that $B^{\triangleright}$ is B-complete. Let $S \in \mathrm{Prof}^{\mathrm{light}}$ and consider the $B^{\triangleright}$-module $\mathcal{M}[S] = \varinjlim_i A_i[S]$. It suffices to show that $\mathcal{M}[S]$ is B-complete, namely, for $C \in \mathscr{D}(B)$ we have

$$R\underline{\mathrm{Hom}}_{B^{\triangleright}}(\mathcal{M}[S], C) = \varprojlim_i R\underline{\mathrm{Hom}}_{A_i}(A_i[S], C) = R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], C).$$

We have to show that $\mathcal{M}[S]$ is $B_{A_i/}$-complete for all i. Let us first argue when I is filtered. Fix $j \in I$, for any $i \geq j$ the module $A_i[S]$ is A_j-complete and taking colimits on i one gets that $\mathcal{M}[S]$ is $B_{A_j/}$-complete. Since the previous hold for all j one deduces that $\mathcal{M}[S]$ is B-complete. Let us now consider a general sifted diagram $\{A_i\}_{i\in I}$. We have then a sifted diagram $\{B_{A_i/}\}_{i\in I}$ of analytic ring structures of $B^{\triangleright}$. Note that the mapping space between two analytic ring structures B' and B'' over $B^{\triangleright}$ is either contractible or empty, depending whether $\mathscr{D}(B'') \subset \mathscr{D}(B')$ or not. Therefore, there is a surective map of categories $\pi : I \to I'$ with I' filtered, such that $\{B_{A_i/}\}_{i\in I}$ can be refined to $\{B_{A_{i'}/}\}_{i'\in I'}$. In particular, for $C \in \mathscr{D}(B^{\triangleright})$ we have

$$C \otimes_{B^{\triangleright}} B = \varinjlim_i C \otimes_{B^{\triangleright}} B_{A_i/}. \tag{4.3}$$

Finally, we get that

$$\begin{aligned}\mathcal{M}[S]\otimes_{B^{\triangleright}} B &= \varinjlim_i (A_i[S]\otimes_{A_i^{\triangleright}} B)\\ &= \varinjlim_i (A_i[S]\otimes_{A_i^{\triangleright}} \varinjlim_j B_{A_j/})\\ &= \varinjlim_i (A_i[S]\otimes_{A_i^{\triangleright}} B_{A_i/})\\ &= \varinjlim_i (A_i \otimes_{A_i^{\triangleright}} (A_i[S]\otimes_{A_i^{\triangleright}} B^{\triangleright}))\\ &= \varinjlim_i (A_i \otimes_{A_i^{\triangleright}} (A_i[S]\otimes_{A_i^{\triangleright}} \varinjlim_j A_j^{\triangleright}))\\ &= \varinjlim_i (A_i \otimes_{A_i^{\triangleright}} (A_i[S]\otimes_{A_i^{\triangleright}} A_i^{\triangleright}))\\ &= \varinjlim_i A_i[S]\\ &= \mathcal{M}[S].\end{aligned}$$

where in the second equality we use (4.3), in the first, third and sixth equalities we use that I is sifted (so the diagonal $I \to I\times I$ is cofinal), and the rest follows from the definitions. □

In order to show that analytic rings admit arbitrary colimits we first need to discuss completions of analytic rings.

Theorem 4.1.12 ([Man22, Proposition 2.3.12])**.** *The functor* $\mathrm{AnRing}\to\mathrm{AnRing}^{uc}$ *has a left adjoint* $A\mapsto A^{=}$ *called the "completion functor". We have* $\mathscr{D}(A)=\mathscr{D}(A)^{=}$ *and* $A^{=,\triangleright} = A\otimes_{A^{\triangleright}} A^{\triangleright}$ *is the* A*-completion of* $A^{\triangleright}$ *(i.e. the unit in* $\mathscr{D}(A)$*). In particular,* AnRing *admits small colimits. A diagram* $\{A_i\}_i$ *of analytic rings has colimit* $B^{=}$ *where* $B=\varinjlim_i A_i$ *is the colimit in the category of uncompleted analytic rings.*

Sketch of the proof. We will prove a weaker version of the theorem where "animated ring" gets replaced by "commutative or $\mathbb{E}_\infty$-ring". Indeed, the difficult part of the theorem is to show that the unit $A^{=,\triangleright}$ has a natural animated ring structure. This will be handled in the next section.

Let B be an analytic ring and A an uncomplete analytic ring. By definition, $\mathrm{Map}_{\mathrm{AnRing}^{un}}(A,B)$ is the full subanima of maps $\mathrm{Map}_{\mathrm{CAlg}(\mathscr{D}(\mathrm{Cond}))}(A^{\triangleright},B^{\triangleright})$ of commutative condensed algebras such that the forgetul functor

$$\mathscr{D}(B^{\triangleright})\to\mathscr{D}(A^{\triangleright})$$

sends $\mathscr{D}(B)$ to $\mathscr{D}(A)$. By [Lur17, Corollary 4.8.5.21] the space $\mathrm{Map}_{\mathrm{CAlg}(\mathscr{D}(\mathrm{Cond}))}(A^{\triangleright},B^{\triangleright})$ is naturally equivalent to the space of $\mathscr{D}(\mathrm{CondAb})$-linear symmetric monoidal functors $\mathscr{D}(A^{\triangleright})\to\mathscr{D}(B^{\triangleright})$. Therefore, $\mathrm{Map}_{\mathrm{AnRing}^{un}}(A,B)$ gets identified with the full subcategory of symmetric monoidal functors as above that factor through

$$\begin{array}{ccc}\mathscr{D}(A^{\triangleright}) & \xrightarrow{B^{\triangleright}\otimes_{A^{\triangleright}}} & \mathscr{D}(B^{\triangleright})\\ \downarrow{\scriptstyle A\otimes_{A^{\triangleright}}} & & \downarrow{\scriptstyle B\otimes_{B^{\triangleright}}}\\ \mathscr{D}(A) & \longrightarrow & \mathscr{D}(B).\end{array}$$

Since both $\mathscr{D}(A)$ and $\mathscr{D}(B)$ are localizations of $\mathscr{D}(A^{\triangleright})$ and $\mathscr{D}(B^{\triangleright})$ respectively, the space $\mathrm{Map}_{\mathrm{AnRing}^{un}}(A,B)$ is naturally equivalent to the space of $\mathscr{D}(\mathrm{CondAb})$-linear symmetric monoidal functors $\mathscr{D}(A)\to\mathscr{D}(B)$, which is also clearly equivalent to $\mathrm{Map}_{\mathrm{AnRing}}(A^{=},B)$, proving the desired adjunction.

The last claim about the computation of the colimit of analytic rings follows directly from the existence of the left adjoint $(-)^{=}$. □

4.2. **Completions of analytic rings.** In this section we will complete the proof of Theorem 4.1.12. For this we need to recall how animated rings are constructed out of connective modules.

Definition 4.2.1. Let $\mathscr{C}$ be a (presentable) compactly projective generated 1-category. Let $\mathscr{C}^0 \subset \mathscr{C}$ be the full subcategory of compact projective objects . The animation of $\mathscr{C}$ (or its non-abelian derived category) is defined as the sifted ind-completion of $\mathscr{C}^0$: $\mathrm{Ani}(\mathscr{C}) := \mathrm{sInd}(\mathscr{C}^0)$ (also denote as $\mathcal{P}_\Sigma(\mathscr{C}^0)$ in [Lur09, §5.5.8]). More precisely, it is the full subcategory

$$\mathrm{sInd}(\mathcal{C}^0) \subset \mathrm{Fun}(\mathcal{C}^{0,\mathrm{op}}, \mathrm{Ani})$$

of accessible presheaves F preserving finite products (i.e. $F(X \sqcup Y) = F(X) \times F(Y)$ for $X, Y \in \mathscr{C}^0$).

Example 4.2.2. Standard examples of animation are the following:

(1) If $\mathscr{C} = \mathrm{Sets}$ is the category of sets then $\mathscr{C}^0$ is the category of finite sets and $\mathrm{Ani}(\mathscr{C}) = \mathrm{Ani}$ is the category of anima or of "spaces".
(2) If $\mathscr{C} = \mathrm{Ab}$ is the category of abelian groups then $\mathscr{C}^0$ is the category of free abelian groups and $\mathrm{Ani}(\mathscr{C})$ is the category of animated abelian groups (also known in the literature as "simplicial abelian groups"). Thanks to the Dold-Kan-correspondence [Lur17, Theorem 1.2.3.7] it is also equivalent to the category $\mathscr{D}_{\geq 0}(\mathbb{Z})$ of connective objects in the ∞-derived category of abelian groups.
(3) If $\mathscr{C} = \mathrm{Ring}$ then $\mathscr{C}^0$ is the category of retracts of polynomial rings in finitely many variables and $\mathrm{Ani}(\mathscr{C}^0)$ is the category AniRing of animated commutative rings (also known as the category of "simplicial commutative rings" in the literature).

Definition 4.2.3 (Symmetric functors)**.** Consider $\mathscr{D}_{\geq 0}(\mathbb{Z})$ the infinity category of animated abelian groups. The symmetric power functors

$$\mathrm{Sym}^n : \mathscr{D}_{\geq 0}(\mathbb{Z}) \to \mathscr{D}_{\geq 0}(\mathbb{Z})$$

are defined as the left derived functors of the usual symmetric power functors in static rings and abelian groups. More explicitly, it is the unique functor preserving sifted colimits and mapping a finite free abelian group F to its symmetric power $\mathrm{Sym}^n F$.

The importance of the symmetric functors for us is that they appear in the monad defining animated rings.

Proposition 4.2.4. *Let* AniRing *be the ∞-category of animated commutative rings. Let $\mathscr{D}_{\geq 0}(\mathbb{Z})$ be the ∞-category of connective abelian group. Then the forgetful functor*

$$G : \mathrm{AniRing} \to \mathscr{D}_{\geq 0}(\mathbb{Z})$$

has a left adjoint given by the left derived functor of the functor $\mathrm{Ab} \to \mathrm{Ring}$ *mapping an abelian group M to its symmetric algebra* $\mathrm{Sym}^\bullet M$*. Furthermore, the previous adjunction is monadic.*

Proof. The forgetful functor $F : \mathrm{Ring} \to \mathrm{Ab}$ has by left adjoint the symmetric power functor $\mathrm{Sym}^\bullet : \mathrm{Ab} \to \mathrm{Ring}$. Let $\mathrm{Ab}^0 \subset \mathrm{Ab}$ and $\mathrm{Ring}^0 \subset \mathrm{Ring}$ denote the full subcategories of compact projective objects, namely, Ab^0 is the category of finite free abelian groups and Ring^0 is the category of (retracts of) polynomial algebras of finite type. The symmetric power functor $\mathrm{Sym}^\bullet$ restricts to a coproduct preserving functor $\mathrm{Sym}^\bullet : \mathrm{Ab}^0 \to \mathrm{Ring}^0$. We can then form the sifted ind categories sInd obtaining the left derived functor

$$\mathrm{Sym}^\bullet \colon \mathscr{D}_{\geq 0}(\mathbb{Z}) \cong \mathrm{sInd}(\mathrm{Ab}^0) \to \mathrm{sInd}(\mathrm{Ring}^0) = \mathrm{AniRing}\,. \tag{4.4}$$

By construction $\mathrm{Sym}^\bullet$ preserves coproducts when restricted to Ab^0, namely, if F_1 and F_2 are finite free abelian groups then $\mathrm{Sym}^\bullet(F_1 \oplus F_2) = \mathrm{Sym}^\bullet F_1 \otimes \mathrm{Sym}^\bullet F_2$. Then, Proposition [Lur09, Proposition 5.5.8.15] (3) implies that (4.4) preserves colimits. By the adjoint functor theorem [Lur09, Corollary 5.5.2.9] the functor $\mathrm{Sym}^\bullet$ has a right adjoint G. Note that by uniqueness of the adjunction, G restricted to $\mathrm{Ring} \subset \mathrm{AniRing}$ is the forgetful functor $G : \mathrm{Ring} \to \mathrm{Ab} \subset \mathscr{D}_{\geq 0}(\mathbb{Z})$. Then, to see that

G is the "forgetful functor" on the category AniRing it will suffice to show that it commutes with sifted colimits. This follows from the fact that $\mathrm{Sym}^\bullet$ sends compact projective objects to compact projective objects: given a sifted diagram $\{A_i\}_{i\in I}$ in AniRing and $F \in \mathrm{Ab}^0$ we have

$$\begin{aligned}\mathrm{Map}_{\mathscr{D}_{\geq 0}(\mathbb{Z})}(F, G(\varinjlim_i A_i)) &= \mathrm{Map}_{\mathrm{AniRing}}(\mathrm{Sym}^\bullet F, (\varinjlim_i A_i)) \\ &= \varinjlim_i \mathrm{Map}_{\mathrm{AniRing}}(\mathrm{Sym}^\bullet F, (A_i)) \\ &= \varinjlim_i \mathrm{Map}_{\mathscr{D}_{\geq 0}(\mathbb{Z})}(F, GA_i) \\ &= \mathrm{Map}_{\mathscr{D}_{\geq 0}(\mathbb{Z})}(F, \varinjlim_i GA_i)\end{aligned}$$

where the first equivalence is the adjunction, the second follows since $\mathrm{Sym}^\bullet F$ is compact projective in AniRing, the third is another adjunction, and the last follows since F is compact projective in $\mathscr{D}_{\geq 0}(\mathbb{Z})$. This proves that the natural map $\varinjlim_i GA_i \to G(\varinjlim_i A_i)$ is an equivalence.

Finally, to show that the adjunction is monadic, by the Barr-Beck-Lurie theorem [Lur17, Theorem 4.7.3.5] it suffices to see that G is conservative; this is obvious since $\mathrm{Sym}^\bullet$ sends Ab^0 to a set of generators of AniRing. □

Remark 4.2.5. Let $\mathscr{C}$ be an ∞-category with finite limits. Then the adjunction $G\colon \mathrm{AniRing} \to \mathscr{D}_{\geq 0}(\mathbb{Z})$ extends to an adjunction at the level of presheaves on $\mathscr{C}$

$$G\colon \mathrm{PSh}(\mathscr{C}, \mathrm{AniRing}) \to \mathrm{PSh}(\mathscr{C}, \mathscr{D}_{\geq 0}(\mathbb{Z})) : \mathrm{Sym}^\bullet. \tag{4.5}$$

Suppose that $\mathscr{C}$ has in addition a Grothendieck topology $\mathcal{T}$, and for a presentable ∞-category $\mathscr{D}$ let $\widehat{\mathrm{Sh}}(\mathscr{C}, \mathscr{D})$ denote the full subcategory of $\mathscr{D}$-valued hypersheaves of $\mathscr{C}$. Then the adjunction (4.5) restricts to an adjuction

$$\widehat{G}\colon \widehat{\mathrm{Sh}}(\mathscr{C}, \mathrm{AniRing}) \to \widehat{\mathrm{Sh}}(\mathscr{C}, \mathscr{D}_{\geq 0}(\mathbb{Z})) : \mathrm{Sym}^\bullet.$$

Indeed, since G preserves limits it maps the full subcategory $\widehat{\mathrm{Sh}}(\mathscr{C}, \mathrm{AniRing}) \subset \mathrm{PSh}(\mathscr{C}, \mathrm{AniRing})$ to $\widehat{\mathrm{Sh}}(\mathscr{C}, \mathscr{D}_{\geq 0}(\mathbb{Z})) \subset \mathrm{PSh}(\mathscr{C}, \mathscr{D}_{\geq 0}(\mathbb{Z}))$. On the other hand, the inclusion of hypersheaves has by left adjoint the hypercompletion functor

$$(-)^\wedge : \mathrm{PSh}(\mathscr{C}, \mathscr{D}) \to \widehat{\mathrm{Sh}}(\mathscr{C}, \mathscr{D}).$$

Thus, the forgetful functor

$$\widehat{\mathrm{Sh}}(\mathscr{C}, \mathrm{AniRing}) \to \mathrm{PSh}(\mathscr{C}, \mathscr{D}_{\geq 0}(\mathbb{Z}))$$

has by left adjoint the hypercompletion of the symmetric functor, namely, $(\mathrm{Sym}^\bullet)^\wedge$. This restricts to an adjunction

$$\widehat{G}\colon \widehat{\mathrm{Sh}}(\mathscr{C}, \mathrm{AniRing}) \to \widehat{\mathrm{Sh}}(\mathscr{C}, \mathscr{D}_{\geq 0}(\mathbb{Z})) : (\mathrm{Sym}^\bullet)^\wedge.$$

It is clear that $\widehat{G}$ is conservative. Moreover, sifted colimits of objects in $\widehat{G}\colon \widehat{\mathrm{Sh}}(\mathscr{C}, \mathrm{AniRing})$ are taken as hypersheafifications of sifted colimits in presheaves. This shows that $\widehat{G}$ also commutes with sifted colimits, and so it is monadic.

Applying the previous construction to $\mathscr{C} = \mathrm{Prof}^{\mathrm{light}}$ endowed with its natural topology (and dropping further notation in the hypercompletion functor) we get the monadic adjunction

$$G\colon \mathrm{Cond}(\mathrm{AniRing}) \to \mathscr{D}_{\geq 0}(\mathrm{CondAb}) : \mathrm{Sym}^\bullet.$$

After the previous preparations we can now state the key proposition regarding the completion of analytic rings.

Proposition 4.2.6 ([CS20, Proposition 12.26])**.** *Let A be an uncompleted analytic ring. Let $\mathrm{AniRing}_{A/}$ be the category of condensed animated $A^{\triangleright}$-algebras whose underlying module is A-complete. Consider the adjunction*

$$\mathrm{Sym}^{\bullet}_{A^{\triangleright}}, G\colon \mathcal{D}_{\geq 0}(A^{\triangleright}) \to \mathrm{AniRing}_{A^{\triangleright}}\,. \tag{4.6}$$

Then for any map $N \to M$ of $A^{\triangleright}$-modules which induces an equivalence after A-completion the natual map

$$A \otimes_{A^{\triangleright}} \mathrm{Sym}^{\bullet}_{A^{\triangleright}} N \to A \otimes_{A^{\triangleright}} \mathrm{Sym}^{\bullet}_{A^{\triangleright}}$$

is also an equivalence. In particular, the monadic adjunction (4.6) *localizes to a monadic adjuction*

$$\mathrm{Sym}^{\bullet}_{A}, G : \mathcal{D}_{\geq 0}(A) \to \mathrm{AniRing}_{A},$$

where $\mathrm{Sym}^{\bullet}_{A} = A \otimes_{A^{\triangleright}} \mathrm{Sym}^{\bullet}_{A^{\triangleright}}$*.*

Proof. This is [CS20, Lemma 12.27]; its proof consists in studying the Goodwillie derivatives of the polynomial functors Sym^{i} and reduce the statement to the fact that for all prime p the Frobenius $\phi : A \to A/p$ is a morphism of analytic rings. This last statement will be proven in §4.3. □

Corollary 4.2.7. *Let A be an uncompleted analytic ring, then the completion $A^{=}$ of A as $\mathbb{E}_{\infty}$-ring has a natural structure of analytic ring making $A \to A^{=}$ a morphism of analytic rings. In other words, $A^{=,\triangleright} = A[*]$ has a natural structure of condensed animated ring defined by the completed symmetric powers of Proposition 4.2.6 and it is the left adjoint of the natural inclusion* $\mathrm{AnRing} \to \mathrm{AnRing}^{uc}$*.*

Proof. This follows from proposition 4.2.6 and the monadic adjunction of Proposition 4.2.4, see Remark 4.2.5. □

4.3. **Frobenius.** In the proof of Proposition 4.2.6 we used the fact that Frobenius induces a morphism of analytic rings. The goal of this section is to prove this fact (Theorem 4.3.2). The key step is Lemma 4.3.1 comparing the Tate constructions of free modules on light profinite sets with C_p-action.

Lemma 4.3.1 ([CS20, Assumption 12.25])**.** *Let A be an analytic ring. Let S be a light profinite set endowed with a C_p-action and let $S_0 = S^{C_p}$ be the fixed points. Then the natural map*

$$A[S_0]^{tC_p} \to A[S]^{tC_p}$$

is an equivalence, where $(-)^{tC_p}$ is the Tate construction.

Proof. Recall the Tate construction for spectra: let $X \in \mathrm{Sp}$ and let C_p be the cyclic group on p-elements. Suppose that we have an homotopic action of C_p on X, then there is a norm map $\mathrm{Nm} : X_{C_p} \to X^{C_p}$ from the homotopic co-invariants to the invariants. The Tate construction is defined as the cofiber

$$X^{tC_p} := \mathrm{cofib}(X_{C_p} \to X^{C_p}).$$

Now let S be a light profinite set endowed with a C_p-action and let $S_0 = S^{C-p}$ be its fixed points. For a light locally profinite set U with compactification $U \subset T$ and boundary ∂T let $A[\overline{U}] := A[T]/A[\partial T]$ be the A-measures on U vanishing at ∞. Set $U = S \backslash S_0$. The module $A[\overline{U}]$ is independent of the compactification since we have a pushout diagram

$$\begin{array}{ccc} \partial T & \longrightarrow & T \\ \downarrow & & \downarrow \\ \{\infty\} & \longrightarrow & U \cup \{\infty\}. \end{array}$$

It suffices to show that $A[\overline{U}]^{tC_p} = 0$. By Proposition 2.1.5 we can write $U = \bigsqcup_n S'_n$ as a countable disjoint union of light profinite sets. The action of C_p on U is then totally discontinuous not having any fixed point. Then, $U/C_p = \bigsqcup_n S''_n$ is a countable disjoint union of light profinite sets, and by taking pullbacks of such decomposition by the map $U \to U/C_p$ we can write $U \cong \bigsqcup_n C_p \times S''_n$. Therefore, if S'' is a compactification of U/C_p, we see that $C_p \times S''$ is a compactification of U. This shows that

$$A[\overline{U}]^{tC_p} = \mathrm{cofib}(A[C_p \times \partial S'']^{tC_p} \to A[C_p \times S'']^{tC_p}).$$

But for any condensed anima T we have $A[C_p \times T] = A[C_p] \otimes A[T]$ and so has vanishing Tate cohomology $A[C_p \times T]^{tC_p} = 0$, proving what we wanted. □

Theorem 4.3.2 ([CS20, Proposition 12.24])**.** *Let A be an analytic ring and let $A/p = A \otimes_{\mathbb{Z}} \mathbb{F}_p$. Then the Frobenius map $\phi : A^{\triangleright} \to A^{\triangleright}/p$ is a map of analytic rings $\phi : A \to A/p$.*

Proof. This is proven in *loc. cit.* where the only condition needed is Assumption 12.25 which always holds true thanks to Lemma 4.3.1. □

4.4. **Invariance of analytic ring structures.** It is useful for constructions of analytic rings to compare analytic ring structures between morphisms of condensed animated rings. In this section we shall prove that analytic ring structures are "formally étale" in the sense that they are invariant under nilpotent thickenings and higher animated structures. We follow [CS20, Lecture XII Appendix 1]. The first result in this direction if the following theorem that encodes the datum of an analytic ring structure in terms of an abelian category.

Theorem 4.4.1. *Let $A^{\triangleright}$ be a condensed animated ring. Then the set of (uncompleted) analytic ring structures A over $A^{\triangleright}$ is in bijection with full subcategories $\mathscr{C}$ of the abelian category $\mathrm{Mod}(\pi_0(A^{\triangleright}))$ satisfying the following properties:*

1. *$\mathscr{C}$ is stable under all limits, colimits and extensions in $\mathrm{Mod}(\pi_0(A^{\triangleright}))$.*
2. *$\mathscr{C}$ is presentable.*
3. *$\mathscr{C}$ is stable under arbitrary higher direct products $\prod_I^{(n)}$.*
4. *For all $S \in \mathrm{Prof}^{\mathrm{light}}$ and $C \in \mathscr{C}$ the Ext modules $\underline{\mathrm{Ext}}^i_{\mathbb{Z}}(\mathbb{Z}[S], C)$ are in $\mathscr{C}$.*

More precisely, given A an analytic ring structure of $A^{\triangleright}$, the category $\mathscr{C} = \mathscr{D}(A) \cap \mathscr{D}^{\heartsuit}(A^{\triangleright})$ satisfies the conditions (1)-(4) above. Conversely, given a subcategory $\mathscr{C}$ as above then the category $\mathscr{D} \subset \mathscr{D}(A^{\triangleright})$ consisting on those complexes C with cohomology groups in $\mathscr{C}$ defines an analytic ring structure on $A^{\triangleright}$.

In order to prove the theorem let us first show a bijection for localizations with weaker conditions.

Proposition 4.4.2 ([CS20, Proposition 12.19])**.** *Let $A^{\triangleright}$ be a condensed animated ring. The collection of full sub ∞-categories $\mathscr{D} \subset \mathscr{D}_{\geq 0}(A^{\triangleright})$ stable under limits and colimits is in natural bijection with the collection of all full subcategories $\mathscr{C} \subset \mathrm{Mod}(\pi_0(A^{\triangleright})) = \mathscr{D}^{\heartsuit}(A^{\triangleright})$ stable under limits, colimits, extensions and higher derived products, via sending $\mathscr{D}$ to the intersection with $\mathrm{Mod}(\pi_0(A^{\triangleright}))$, and $\mathscr{C}$ to the full subcategory $\mathscr{D}$ of all $C \in \mathscr{D}_{\geq 0}(A^{\triangleright})$ such that $\pi_i(C) \in \mathscr{C}$ for all $i \geq 0$. Moreover, $\mathscr{D}$ is presentable if and only if $\mathscr{C}$ is so.*

Proof. Let $\mathscr{D} \subset \mathscr{D}_{\geq 0}(A^{\triangleright})$ be a full subcategory stable under limits and colimits. Define $\mathscr{C} = \mathscr{D} \cap \mathscr{D}^{\heartsuit}(A^{\triangleright})$. Given $C \in \mathscr{D}$ the functor $\tau_{\geq 1} C$ is the suspension of the loops of C, and so it is in $\mathscr{D}$. This shows that $\pi_0(C)[0] \in \mathscr{C}$ being the cofiber of $\tau_{\geq 1} C[1] \to C$. Then, $\pi_i(C)[0] \in \mathscr{C}$ for all $i \geq 0$. Since $\mathscr{D}$ is stable under finite limits and colimits, this shows that $\mathscr{C}$ is stable under finite limits, finite colimits and extensions. Since arbitrary direct sums are exact and $\mathscr{D}$ has all colimits, then $\mathscr{C}$ has arbitrary direct sums and it is stable under all colimits. Finally, given a family of objects X_i in $\mathscr{C}$, the homotopy product $\prod_i (X_i[n])$ is in $\mathscr{D}$ for all $n \in \mathbb{N}$ as $\mathscr{D}$ is stable under all limits. Taking homotopy groups we see that the higher products $\prod_i^{(n)} X_i$ are in $\mathscr{C}$ for all $n \in \mathbb{N}$. In particular, $\mathscr{C}$ has arbitrary products and so it is stable under all limits.

Conversely, let $\mathscr{C} \subset \mathscr{D}^{\heartsuit}(A^{\triangleright})$ be a full subcategory stable under all limits, colimits, extensions and arbitrary higher products. Let $\mathscr{D} \subset \mathscr{D}_{\geq 0}(A^{\triangleright})$ be the full subcategory consisting on those objects C

with homotopy groups in $\mathscr{C}$. Stability under finite limits and extensions in $\mathscr{C}$ shows that $\mathscr{D}$ is stable under fibers and cofibers. It is also clear that $\mathscr{D}$ is stable under Postnikov limits. Since arbitrary direct sums are exact then $\mathscr{D}$ is stable under direct sums and so under all colimits. Stability under arbitrary higher products in $\mathscr{C}$ implies that $\mathscr{D}$ is stable under arbitrary homotopy products, and so it is stable under all limits.

Finally, if $\mathscr{C}_0$ is a family of generators of $\mathscr{D}$ then its homotopy groups form a family of generators of $\mathscr{C}$. Conversely, given a family of generators of $\mathscr{C}$ all their shifts form a family of generators for $\mathscr{D}$. This proves that $\mathscr{D}$ is presentable if and only if $\mathscr{C}$ is so. □

Remark 4.4.3. There is a minor difference between the statement of Proposition 4.4.2 and [CS20, Proposition 12.19], namely in the former we ask for the stability of higher derived products. In the classical condensed framework arbitrary products are exact thanks to the extremally totally disconnected spaces. However, in the light set up a priori only countable products are exact, and there could be higher derived functors for sets with non countable cardinality.

Proof of Theorem 4.4.1. Given an analytic ring structure A on $A^{\triangleright}$, the category $\mathscr{C} = \mathscr{D}(A) \cap \mathscr{D}^{\heartsuit}(A^{\triangleright})$ satisfies (1)-(3) of Theorem 4.4.1 thanks to Proposition 4.4.2. Moreover, condition (2) of Definition 4.1.1 and Proposition 4.1.7 imply that $\mathscr{C}$ is also stable under internal Ext functors. On the other hand, Proposition 4.1.7 also says that $\mathscr{D}(A) \subset \mathscr{D}$ is the full subcategory consisting on complexes whose cohomology groups are in $\mathscr{C}$.

Conversely, let $\mathscr{C} \subset \mathrm{Mod}(\pi_0(A^{\triangleright}))$ be a full subcategory as in the statement of the theorem and let $\mathscr{D}_{\geq 0} \subset \mathscr{D}_{\geq 0}(A^{\triangleright})$ be the full subcategory of objects whose cohomology groups are in $\mathscr{C}$. Proposition 4.4.2 implies that $\mathscr{D}_{\geq 0}$ is stable under all limits and colimits and that it is presentable. The same holds true for its stabilization $\mathscr{D} \subset \mathscr{D}(A^{\triangleright})$ consisting on all the complexes whose cohomology groups are in $\mathscr{C}$. Since $\mathscr{D}$ is stable under all limits, colimits and is presentable, we have the left adjoint for the inclusion $L : \mathscr{D}(A^{\triangleright}) \to \mathscr{D}$. Moreover, this left adjoint preserves connective objects since $\mathscr{D}_{\geq 0}$ is also stable under all limits and colimits in $\mathscr{D}_{\geq 0}(A^{\triangleright})$. This proves conditions (1) and (3) of Definition 4.1.1. It is left to show that $\mathscr{D}$ is stable under $R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], -)$ for $S \in \mathrm{Prof}^{\mathrm{light}}$. Let $M \in \mathscr{D}$. By writing $M = \varprojlim_n \tau_{\geq n} M$ as limit of its Postnikov tower we can assume that $M \in \mathscr{D}_{\leq 0}$ is co-connective. Then, there is a convergent expectral sequence with second page

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p(\mathbb{Z}[S], \pi_{-q}(M)) \Rightarrow \pi_{-p-q}(R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], M)).$$

Since all the objects in the E_2-page of the spectral sequence are in $\mathscr{C}$ by hypothesis, and since $\mathscr{C}$ is stable under limits, colimits and extensions, one deduces that the cohomology groups of $R\underline{\mathrm{Hom}}(\mathbb{Z}[S], M)$ are in $\mathscr{C}$. We deduce that $\mathscr{D}$ satisfies condition (2) of Definition 4.1.1, and so it defines an analytic ring structure on $A^{\triangleright}$. □

The first application of Theorem 4.4.1 is the homotopy invariance of the analytic ring structures.

Corollary 4.4.4 ([CS20, Proposition 12.21]). *Let $A^{\triangleright} \to B^{\triangleright}$ be a map of animated condensed rings such that $\pi_0(A^{\triangleright}) \to \pi_0(B^{\triangleright})$ is an isomorphism. There is a bijection between (uncompleted) analytic ring structures of $A^{\triangleright}$ and $B^{\triangleright}$ given by mapping A to $B^{\triangleright}_{A/}$.*

Proof. By Theorem 4.4.1 analytic ring structures on $A^{\triangleright}$ are in bijection with suitable localizations of the abelian category $\mathrm{Mod}(\pi_0(A^{\triangleright}))$. This proves the corollary. □

Another application of Theorem 4.4.1 is the invariance of analytic ring structures under nilpotent thickenings.

Proposition 4.4.5 ([CS20, Proposition 12.23]). *Let $A^{\triangleright} \to B^{\triangleright}$ be a map of condensed animated rings such that the kernel I of $\pi_0(A^{\triangleright}) \to \pi_0(B^{\triangleright})$ is nilpotent. Then there is a natural bijection of uncompleted analytic ring structures on $A^{\triangleright}$ and $B^{\triangleright}$ mapping an analytic ring structure A of $A^{\triangleright}$ to the induced analytic ring structure $B^{\triangleright}_{A/}$.*

Proof. By Proposition 4.4.4 we can assume that $A^{\triangleright}$ and $B^{\triangleright}$ are static rings. By induction, we can even assume that $I^2 = 0$.

Let B be an analytic ring structure on $B^{\triangleright}$ corresponding to some category $\mathcal{C}_B$. Let $\mathscr{C} \subset \mathscr{D}^{\heartsuit}(A^{\triangleright})$ be the full subcategory of objects M such that IM and M/IM are in $\mathscr{D}(B)$. The category $\mathscr{C}$ is clearly presentable. We claim that it is stable under limits, colimits, extensions, arbitrary higher direct products and internal Ext groups from condensed abelian groups. It is clear from the definition that $\mathscr{C}$ is stable under kernels, cokernels and extensions and that it contains $\mathscr{C}_B$. It also contains arbitrary direct sums as they are exact. To see that it contains arbitrary higher products consider a family of objects $\{M_i\}_{i\in I}$ in $\mathscr{C}$. We then have a fiber sequence of homotopy products

$$\prod_i^h IM_i \to \prod_i^h M_i \to \prod_i^h M_i/IM_i.$$

Taking the long exact complex we see that the higher product $\prod^{(n)} M_i$ are in $\mathscr{C}$, namely, by Theorem 4.4.1 we know that arbitrary higher products of objects in $\mathscr{C}_B$ stay in $\mathscr{C}_B$. Finally, stability under internal Ext of $\mathscr{C}_B$ with condensed abelian groups follows by the long exact sequence induced by the short exact sequence $0 \to IM \to M \to M/IM \to 0$. ☐

4.5. **Morphisms of analytic rings.** Let A and B be analytic rings and let $f : A^{\triangleright} \to B^{\triangleright}$ be a morphism of condensed animated rings. We want to have a criterion for the map f to be a morphism of analytic rings $f : A \to B$. The category $\mathscr{D}_{\geq 0}(B)$ is generated by the objects $B[S]$ for S a light profinite set. Then, f is a morphism of analytic rings if and only if $B[S]$ is A-complete for all S-light profinite. Suppose that instead we are given with functorial maps $A[S] \to B[S]$ linear over $A^{\triangleright} \to B^{\triangleright}$ commuting with the map from $S \in \mathrm{Prof}^{\mathrm{light}}$. Then this datum produces a map of analytic ring under a mild condition:

Proposition 4.5.1 ([CS19, Proposition 7.14]). *Keep the previous notation. Suppose that for all $S \in \mathrm{Prof}^{\mathrm{light}}$ with a map $S \to A^{\triangleright}$, inducing $A[S] \to A[*]$ in $\mathscr{D}(A^{\triangleright})$, and from the composite $S \to A^{\triangleright} \to B^{\triangleright}$, a unique map $B[S] \to B[*]$ in $\mathscr{D}(B^{\triangleright})$, the diagram*

$$\begin{array}{ccc} \pi_0(A[S]) & \longrightarrow & \pi_0(A[*]) \\ \downarrow & & \downarrow \\ \pi_0(B[S]) & \longrightarrow & \pi_0(B[*]) \end{array}$$

commutes. Then $f : A^{\triangleright} \to B^{\triangleright}$ is a morphism of analytic rings $f : A \to B$.

Proof. Let $\mathscr{C}_A$ and $\mathscr{C}_B$ be the hearts of the categories of complete A and B-modules respectively. By Theorem 4.4.1 it suffices to show that objects in $\mathscr{C}_B$ are in $\mathscr{C}_A$ when seen as $A^{\triangleright}$-modules. Since the objects $\pi_0(\mathcal{B}[S])$ are generators of $\mathscr{C}_B$ it suffices to prove that they are in $\mathscr{C}_A$. This reduces the question to the abelian situation, where the proof of [CS19, Proposition 7.14] applies. ☐

4.6. **Localizing by killing algebras.** In the "old" foundations of condensed mathematics the construction of analytic rings was a big challenge. The construction of the solid integers required a full understanding of extension groups of locally compact abelian groups, and the construction of the liquid rings involved a lot of non-locally convex functional analysis. In the new framework of light condensed mathematics it is much easier to construct analytic rings out from the internally compact projective object P of null sequences. This simplifies the construction of solid rings, and gives a natural construction of gaseous rings motivated from the Tate curve. A disclaimer: the construction of the liquid rings remains as difficult as before and a priori the light theory does not help to simplify its construction. Nevertheless, we can now construct localization of categories of modules in a much more general way as we shall explain down below.

Let $\mathscr{C}$ be a presentably symmetric monoidal stable ∞-category. Let $A \in \mathscr{C}$ be an object endowed with the following two maps

(1) $m\colon A\otimes A\to A$.
(2) $\mu\colon 1\to A$

such that the composite $A\xrightarrow{\mu\otimes \mathrm{id}_A} A\otimes A\xrightarrow{m} A$ is the identity of A. We let $\mathscr{D}\subset\mathscr{C}$ be the full subcategory of objects M such that $\underline{\mathrm{Hom}}(A,M)=0$. It is clear that $\mathscr{D}$ is presentable, and that it is stable under limits in $\mathscr{C}$ and under internal Hom.

Our goal is to construct explicitly the localization functor $\mathscr{C}\to\mathscr{D}$. Let $C=\mathrm{fib}(1\to A)$ be the fiber. Define $F:\mathscr{C}\to\mathscr{C}$ to be the functor $\underline{\mathrm{Hom}}(C,-)$. Since we have a map $C\to 1$, there is a natural transformation of functors $\mathrm{id}_{\mathscr{C}}\to F$.

Lemma 4.6.1. *Let $X\in\mathscr{C}$ and $M\in\mathscr{D}$. Then $\underline{\mathrm{Hom}}(F(X),M)\to\underline{\mathrm{Hom}}(X,M)$ is an equivalence.*

Proof. By unraveling the constructions, it suffices to show that

$$\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A,X),M)=0. \tag{4.7}$$

We claim that $\underline{\mathrm{Hom}}(A,X)$ is a retract of $A\otimes\underline{\mathrm{Hom}}(A,X)$. Suppose the claim holds, then we get that

$$\underline{\mathrm{Hom}}(A\otimes\underline{\mathrm{Hom}}(A,X),M)=\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A,X),\underline{\mathrm{Hom}}(A,M))=0,$$

which implies the vanishing of (4.7). Let us now prove the claim. The multiplication map $m: A\otimes A\to A$ induces a map

$$\underline{\mathrm{Hom}}(A,X)\to\underline{\mathrm{Hom}}(A\otimes A,X)$$

which is adjoint to a map

$$A\otimes\underline{\mathrm{Hom}}(A,X)\to\underline{\mathrm{Hom}}(A,X).$$

On the other hand, the unit map $\mu:1\to A$ induces a map $\underline{\mathrm{Hom}}(A,X)\to A\otimes\underline{\mathrm{Hom}}(A,X)$. Then a diagram chasing shows that the composite

$$\underline{\mathrm{Hom}}(A,X)\to A\otimes\underline{\mathrm{Hom}}(A,X)\to\underline{\mathrm{Hom}}(A,X) \tag{4.8}$$

is the identity map, proving the claim. Indeed, the diagram (4.8) is adjoint to a diagram

$$f\colon A\otimes\underline{\mathrm{Hom}}(A,X)\xrightarrow{\mu\otimes\mathrm{id}_A} A\otimes A\otimes\underline{\mathrm{Hom}}(A,X)\xrightarrow{g} X$$

where g is the composite

$$g\colon A\otimes A\otimes\underline{\mathrm{Hom}}(A,X)\xrightarrow{m^*} A\otimes A\otimes\underline{\mathrm{Hom}}(A\otimes A,X)\xrightarrow{ev_{A\otimes A}} X.$$

Then, we have a commutative square

$$\begin{array}{ccc} A\otimes A\otimes\underline{\mathrm{Hom}}(A,X) & \xrightarrow{m^*} & A\otimes A\otimes\underline{\mathrm{Hom}}(A\otimes A,X) \\ \downarrow{\scriptstyle m\otimes\mathrm{id}} & & \downarrow{\scriptstyle ev_{A\otimes A}} \\ A\otimes\underline{\mathrm{Hom}}(A,X) & \xrightarrow{ev_A} & X \end{array}$$

Therefore, f is also the composite

$$A\otimes\underline{\mathrm{Hom}}(A,X)\xrightarrow{\mu\otimes\mathrm{id}_A} A\otimes A\underline{\mathrm{Hom}}(A,X)\xrightarrow{m\otimes\mathrm{id}} A\otimes\underline{\mathrm{Hom}}(A,X)\xrightarrow{ev_A} X$$

which is the same as the evaluation map ev_A since $m\circ(\mu\otimes\mathrm{id}_A)=\mathrm{id}_A$. Taking adjoints, one deduces that the composite (4.8) is the identity. $\square$

Let $n\in\mathbb{N}$ and let $F^n:\mathscr{C}\to\mathscr{C}$ be the n-th iteration of the functor F. The natural transformation $\mathrm{id}_{\mathscr{C}}\to F$ produces a sequential diagram of natural transformations

$$\mathrm{id}_{\mathscr{C}}\to F\to F^2\to\cdots\to F^n\to\cdots.$$

We let $F^\infty=\varinjlim_n F^n$. Lemma 4.6.1 shows that for all $n\in[0,\infty]$, $X\in\mathscr{C}$ and $M\in\mathscr{D}$ the natural map

$$\underline{\mathrm{Hom}}(F^n(X),M)\to\underline{\mathrm{Hom}}(X,M) \tag{4.9}$$

is an equivalence. We want to impose some conditions on F for F^∞ to be a left adjoint of the inclusion.

Proposition 4.6.2. *Suppose that one of the following conditions hold:*

(1) *The sequential colimit $F^\infty(X) = \varinjlim_n F^n(X)$ stabilizes for all X (eg. if A is idempotent).*
(2) *$\underline{\mathrm{Hom}}(A, -) : \mathscr{C} \to \mathscr{C}$ commutes with sequential colimits (eg. if A is internally compact in $\mathscr{C}$).*

Then $F^\infty : \mathscr{C} \to \mathscr{C}$ lands in $\mathscr{D}$ and is the left adjoint of the inclusion $\mathscr{D} \subset \mathscr{C}$.

Proof. By (4.9) it suffices to show that F^∞ lands in $\mathscr{D}$. Conditions (1) and (2) imply that for all $X \in \mathscr{C}$ the natural map

$$\varinjlim_n \underline{\mathrm{Hom}}(A, F^n(X)) \to \underline{\mathrm{Hom}}(A, F^\infty(X))$$

is an equivalence. Note that we have a commutative diagram whose rows are fiber sequences

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(A, F^{n+1}(X)) & \longrightarrow & F^{n+1}(X) & \longrightarrow & F^{n+2}(X) \\
\uparrow & & \uparrow & \overset{\mathrm{id}}{\nwarrow} & \uparrow \\
\underline{\mathrm{Hom}}(A, F^{n}(X)) & \longrightarrow & F^{n}(X) & \longrightarrow & F^{n+1}(X).
\end{array}$$

Then, taking colimits as $n \to \infty$ in the columns, we obtain a fiber sequence

$$\varinjlim_n \underline{\mathrm{Hom}}(A, F^n(X)) \to \varinjlim_n F^n(X) \xrightarrow{\sim} \varinjlim_n F^{n+1}(X),$$

where the right arrow is an equivalence. This proves that $\underline{\mathrm{Hom}}(A, F^\infty(X)) = \varinjlim_n \underline{\mathrm{Hom}}(A, F^n(X)) = 0$ as wanted. $\square$

Example 4.6.3. Some classical localizations in commutative algebra appear in the form of Proposition 4.6.2.

(1) Let R be an animated ring, $\mathscr{C} = \mathscr{D}(R)$ and $P = R/^{\mathbb{L}} f$. Then the category $\mathscr{D} \subset \mathscr{C}$ of objects M such that $R\underline{\mathrm{Hom}}_R(P, M) = 0$ is precisely $\mathscr{D} = \mathscr{D}(R[1/f])$. Indeed, an explicit computation shows that $F^\infty(M) = \varinjlim_{\times f} M = M[1/f]$.
(2) Let us keep R and $\mathscr{C}$ as before and take $P = R[1/f]$. Then P is an idempotent algebra and the category $\mathscr{D} \subset \mathscr{C}$ of R-modules M such that $R\underline{\mathrm{Hom}}_R(P, M) = 0$ is precisely the category of f-adicaly complete modules. The functor F^∞ stabilizes for $n = 1$ and $F(M) = \varprojlim_n M/^{\mathbb{L}} f^n$ is the f-adic completion functor.

Example 4.6.4. Let $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ be the free condensed abelian group of null sequences. By Proposition 3.1.3 it has a natural algebra structure making $\mathbb{Z}[q] \to P$ a morphism of algebras, where q is mapped to $[0]$. We will write $P = \mathbb{Z}[\widehat{q}]$.

(1) The multiplication by q in $\mathbb{Z}[\widehat{q}]$ corresponds to the shift map $\mathrm{Shift} : P \to P$. Then, the category of solid abelian groups is precisely the category of those condensed abelian groups M such that
$$\underline{\mathrm{Hom}}(\mathbb{Z}[\widehat{q}]/(1-q), M) = 0.$$
The object P is internally compact projective, then the previous localization lands in the case (2) of Proposition 4.6.2.
(2) Let $\mathbb{Z}_\square$ be the ring of solid integers. We know that $\mathbb{Z}_\square \otimes_{\mathbb{Z}} \mathbb{Z}[\widehat{q}] = \mathbb{Z}[[q]]$ is a power series ring in the variable q. We can construct additional solid structures arising from polynomial algebras as follows: we define the category of solid $\mathbb{Z}[T]_\square$-modules, denoted by $\mathrm{Mod}(\mathbb{Z}[T]_\square)$ to be the full subcategory of $\mathbb{Z}$-solid $\mathbb{Z}[T]$-modules M such that
$$\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[q]][T]/(1-Tq), M) = 0.$$

Heuristically, we are asking for a null sequence $(b_n)_{n\in\mathbb{N}}$ to be T-summable, i.e. for $\sum_n b_n T^n$ to converge. Note that $\mathbb{Z}[[q]][T]/(1-qT) = \mathbb{Z}((T^{-1}))$ is the ring of Laurent power series in T^{-1}. By Example (3.5.1) the algebra $\mathbb{Z}((T^{-1}))$ is idempotent over $\mathbb{Z}[T]$. Then the previous localization lands in both conditions (1) and (2) of Proposition 4.6.2.

(3) The new kind of analytic rings that can be constructed abstractly using Proposition 4.6.2 are the gaseous rings. Let $A^{\triangleright} = \mathbb{Z}[\widehat{q}][q^{-1}]$ and consider the algebra $A^{\triangleright}\otimes_{\mathbb{Z}} P$. Let T denote the variable of P. Then the gaseous structure over $A^{\triangleright}$ is the localization with respect to the algebra $A\otimes_{\mathbb{Z}} P/(1-qT)$. In other words, an object $M\in\mathscr{D}(A^{\triangleright})$ is gaseous if

$$\underline{\mathrm{Hom}}_{A^{\triangleright}}(A\otimes_{\mathbb{Z}} P/(1-qT), M) = 0.$$

(4) More generally, given an analytic ring A consider $P_A = A\otimes_{\mathbb{Z}}\mathbb{Z}[\widehat{q}]$. Then, for any P_A-algebra R which is a perfect P_A-module one can consider the localization $\mathscr{D}\subset\mathscr{D}(A)$ consisting on the objects M such that $\underline{\mathrm{Hom}}(R,M)=0$. The category $\mathscr{D}$ satisfies conditions (1) and (2) of Definition 4.1.1. The only constrain to define an analytic ring structure for $A^{\triangleright}$ is the connectivity condition (3). Nevertheless, this solves the problem of constructing several examples of analytic rings by a systematic procedure (after verifying condition (3) for connectivity).

5. Solid analytic rings

In this section we give examples of analytic rings arising from non-archimedean geometry. In Section 3 we constructed the analytic ring $\mathbb{Z}_{\square}$ of solid integers, our first objective is to generalize this construction to finite type algebras over $\mathbb{Z}$ and then to arbitrary discrete animated rings. We continue with the definition of solid quasi-coherent sheaves for schemes.

5.1. **Smashing spectrum.** Before giving examples of solid analytic rings let us discuss some general constructions in (stable) symmetric monoidal ∞-categories that will be crucial in the definition of analytic stacks. These are natural categorifications of open and closed immersions of spaces from the point of view of a six functor formalism. We shall follow [CS22, Lecture V], see also [Aok23].

5.1.1. *Topological six functors.* As motivation let us recall some basic facts about sheaves on topological spaces. Let X be a topological space, and let $D(X,\mathbb{Z})$ be the derived category of abelian sheaves on X. Given $U\subset X$ an open subspace and $Z = X\backslash U$ the closed complement we have different functors relating the categories $D(U,\mathbb{Z})$, $D(Z,\mathbb{Z})$ and $D(X,\mathbb{Z})$. More precisely, we have

- Pullback functors
$$\iota^*: D(X,\mathbb{Z})\to D(Z,\mathbb{Z})$$
$$j^*: D(X,\mathbb{Z})\to D(U,\mathbb{Z}).$$
- The functor ι^* has a right adjoint given by a pushforward or extension by 0 functor
$$\iota_*: D(Z,\mathbb{Z})\to D(X,\mathbb{Z}).$$
- ι_* itself has also a right adjoint given by sections supported at Z
$$\iota^!: D(X,\mathbb{Z})\to D(Z,\mathbb{Z}).$$

The functor ι_* satisfies the projection formula, namely, the following arrow is an equivalence

$$\iota_* N\otimes_{\mathbb{Z}_X} M \xrightarrow{\sim} \iota_*(N\otimes_{\mathbb{Z}_Z}\iota^* M). \tag{5.1}$$

More explicitly, $\iota_*\mathbb{Z}_Z$ is the locally constant sheaf $\mathbb{Z}$ supported on Z, and we have

$$\iota_*\iota^* = \iota_*\mathbb{Z}_Z\otimes_{\mathbb{Z}_X} -.$$

The functor $\iota^!$ is then described as

$$\iota_*\iota^! = R\underline{\mathrm{Hom}}_{\mathbb{Z}_X}(\iota_*\mathbb{Z}_Z, -).$$

On the other hand, the functor j^* has both left and right adjoints.

- The right adjoint $j_* : D(U,\mathbb{Z}) \to D(X,\mathbb{Z})$ is the natural pushforward functor.
- The left adjoint $j_! : D(U,\mathbb{Z}) \to D(X,\mathbb{Z})$ is the natural extension by 0 functor.

Furthermore, the functor $j_!$ satisfies the projection formula, namely, the natural map

$$j_!(N \otimes_{\mathbb{Z}_U} M) \xrightarrow{\sim} j_! N \otimes_{\mathbb{Z}_X} M \tag{5.2}$$

is an equivalence.

It turns out that the functors ι_*, j_* and $j_!$ are fully faithful; this is a consequence of the fact that the map $\mathbb{Z}_X \to \iota_* \mathbb{Z}_Z$ is idempotent in $D(X,\mathbb{Z})$, namely, that the natural map $\iota_*\mathbb{Z}_Z \to \iota_*\mathbb{Z}_Z \otimes^L_{\mathbb{Z}_X} \iota_*\mathbb{Z}_Z$ is an equivalence. Hence, one has natural exact triangles in $D(X,\mathbb{Z})$ involving all four functors:

$$\iota_*\iota^! \to \mathrm{id}_X \to j_*j^*$$
$$j_!j^* \to \mathrm{id}_X \to \iota_*\iota^*.$$

The previous exact triangles give rise to "Verdier exact sequences" of derived categories

$$D(Z,\mathbb{Z}) \underset{\overset{\longleftarrow}{\iota^!}}{\overset{\overset{\iota^*}{\longleftarrow}}{\xrightarrow{\iota_*}}} D(X,\mathbb{Z}) \underset{\overset{\longleftarrow}{j_*}}{\overset{\overset{j_!}{\longleftarrow}}{\xrightarrow{j^*}}} D(U,\mathbb{Z}). \tag{5.3}$$

One would like to generalize the localization sequences of (5.3) altogether with the projection formulas (5.1) and (5.2) in order to talk about abstract open and closed immersions of symmetric monoidal categories. This idea is realized thanks to the smashing spectrum.

5.1.2. *Smashing spectrum and idempotent algebras.* For a general notion of open and closed immersions the key objects are idempotent algebras.

Definition 5.1.1. Let $\mathscr{C}$ be a presentably symmetric monoidal stable ∞-category with unit 1. In particular, $\mathscr{C}$ is closed, i.e., it has an internal $\underline{\mathrm{Hom}}$.

An idempotent algebra in $\mathscr{C}$ is a map

$$\mu : 1 \to A$$

in $\mathscr{C}$ such that the natural map

$$A \xrightarrow{\mathrm{id}_A \otimes \mu} A \otimes A$$

is an equivalence. A morphism $A \to B$ of idempotent algebras is a map in $\mathscr{C}$ preserving the unit. We let $\mathscr{S}(\mathscr{C})$ be the opposite of the category of idempotent algebras in $\mathscr{C}$ and call it the *smashing spectrum of* $\mathscr{C}$.

A priori an idempotent algebra is not required to have any algebra structure. However it will be always endowed with a natural commutative algebra structure arising from the natural equivalences

$$A \cong A \otimes A \cong A \otimes A \otimes A \cong \cdots$$

together with all the higher coherences, see [Lur17, Proposition 4.8.2.9].

Given A an idempotent algebra in $\mathscr{C}$ the natural map

$$\mathrm{Mod}_A(\mathscr{C}) \to \mathscr{C}$$

is fully faithful, namely, $M \in \mathscr{C}$ is an A-module if and only if $M \xrightarrow{\mu \otimes \mathrm{id}_M} A \otimes M$ is an equivalence. In other words, being an A-module for an object in $\mathscr{C}$ is a property and not additional structure. One also has that M is an A-module if and only if $\underline{\mathrm{Hom}}(A, M) \to M$ is an isomorphism.

Finally, given A and B idempotent algebras, the mapping space $\mathrm{Map}_{1/}(A, B)$ of idempotent algebras is either contractible or empty. This shows that $\mathscr{S}(\mathscr{C})$ is a poset. Actually, the category $\mathscr{S}(\mathscr{C})$ has the structure of a *locale*, i.e. it behaves as the poset of closed subspaces of a topological space:

Proposition 5.1.2 ([CS22, Proposition 5.3], [Aok23, Theorem 3.8]). *The poset $\mathscr{S}(\mathscr{C})$ is a locale whose closed subspaces $Z \subset \mathscr{S}(\mathscr{C})$ correspond to idempotent algebras A of $\mathscr{C}$, so that*

(1) *$Z \cap Z'$ corresponds to $A \otimes A'$;*
(2) *$Z \subset Z'$ if and only if $A \otimes A' = A$;*
(3) *$Z \cup Z'$ corresponds to $\mathrm{fib}(A \oplus A' \to A \otimes A')$, the unit is given by $1 \xrightarrow{\mu_A \oplus (-\mu_{A'})} A \oplus A'$. Equivalently, $Z \cup Z'$ corresponds to the pullback $A \times_{A \otimes A'} A'$.*
(4) *$\bigcap_i A_i$ corresponds to $\varinjlim_i A_i$.*

We will often let $Z \subset \mathscr{S}(\mathscr{C})$ denote a closed subspace of the local of $\mathscr{C}$ and let $A(Z)$ be the attached idempotent algebra.

5.1.3. *Open and closed immersions of symmetric monoidal categories.* Let Sym be the ∞-category of presentably symmetric monoidal stable ∞-categories. We will take inspiration from the six functors of topological spaces to define open and closed immersions. For geometric reasons we will work with the opposite category $\mathrm{Sym}^{\mathrm{op}}$, given an arrow $f : \mathscr{C} \to \mathscr{D}$ in $\mathrm{Sym}^{\mathrm{op}}$ we shall write $f^* : \mathscr{D} \to \mathscr{C}$ for the corresponding map in Sym.

Definition 5.1.3. Let $f : \mathscr{C} \to \mathscr{D}$ be a map in $\mathrm{Sym}^{\mathrm{op}}$.

(1) We say that f is a closed immersion if f^* has a colimit preserving fully faithful right adjoint $f_* : \mathscr{C} \to \mathscr{D}$ such that for $M \in \mathscr{C}$ and $N \in \mathscr{D}$ the natural map
$$f_*M \otimes N \to f_*(M \otimes f^*N)$$
is an equivalence.
(2) We say that f is an open immersion if f^* has a fully faithful left adjoint $f_! : \mathscr{C} \to \mathscr{D}$ such that for $M \in \mathscr{C}$ and $N \in \mathscr{D}$ the natural map
$$f_!(M \otimes f^*N) \to f_!M \otimes N$$
is an equivalence.

In a few words, a morphism $f : \mathscr{C} \to \mathscr{D}$ in $\mathrm{Sym}^{\mathrm{op}}$ is a closed immersion if and only if f_* is colimit preserving, fully faithful and satisfies the projection formula (so we shall have $f_! = f_*$). Similarly, f is an open immersion if and only iff f^* has a left adjoint $f_!$ which is fully faithful and satisfies projection formula (so we shall have $f^* = f^!$). The following proposition characterizes closed and open immersions in terms of the smashing spectrum.

Proposition 5.1.4 ([CS22, Proposition 6.5]). *Let $f : \mathscr{C} \to \mathscr{D}$ be a morphism in $\mathrm{Sym}^{\mathrm{op}}$.*

(1) *f is a closed immersion if and only if there is a (necessarily unique) idempotent algebra $A \in \mathscr{D}$ such that $1_{\mathscr{C}} \to f^*A$ is an equivalence, and the induced natural map $\mathrm{Mod}_A(\mathscr{D}) \to \mathscr{C}$ is an equivalence.*
(2) *f is an open immersion if and only if there is a (necessarily unique) idempotent algebra A such that $f^*A = 0$, and the induced natural map $\mathscr{D}/\mathrm{Mod}_A(\mathscr{D}) \to \mathscr{C}$ is an equivalence.*

Remark 5.1.5. Let $\mathscr{C} \in \mathrm{Sym}$ and let $A \in \mathscr{C}$ be an idempotent algebra with associated closed subspace $Z \subset \mathscr{S}(\mathscr{C})$ and open complement U. Let $\mathscr{C}(Z) = \mathrm{Mod}_A(\mathscr{C})$ and $\mathscr{C}(U) = \mathscr{C}/\mathscr{C}(U)$ denote the closed and open localizations associated to Z and U respectively. For future reference we shall write explicitly the six functors for open and closed immersions.

(1) The pullback map $\iota^* : \mathscr{C} \to \mathscr{C}(Z)$ is given by the base change $\iota^*M = A \otimes M$ for $M \in \mathscr{C}$. The pushforward $\iota_* : \mathscr{C}(Z) \to \mathscr{C}$ is the forgetful functor and the $\iota^!$ is determined by
$$\iota_*\iota^!M = \underline{\mathrm{Hom}}_{\mathscr{C}}(A, M)$$
for $M \in \mathscr{C}$.

(2) We let $j^* : \mathscr{C} \to \mathscr{C}(U)$ denote the pullback or localization map. The functors $j_!$ and j_* are determined by

$$j_! j^* M = \mathrm{fib}(1 \to A) \otimes M$$

and

$$j_* j^* = \underline{\mathrm{Hom}}_{\mathscr{C}}(\mathrm{fib}(1 \to A), M)$$

for $M \in \mathscr{C}$.

Closed and open immersions behave topologically as expected along pullbacks in $\mathrm{Sym}^{\mathrm{op}}$, inclusions, unions and intersections (see Lemma 6.4 and Corollary 6.6 of [CS22]). Furthermore, one gets "for free" the descent of the underlying symmetric monoidal categories along the topology of the locales.

Theorem 5.1.6 ([CS22, Theorem 6.7])**.** (1) *There is a Grothendieck topology in* $\mathrm{Sym}^{\mathrm{op}}$ *where the coverings of* $\mathscr{C}$ *are given by open localizations of* $\mathscr{C}$ *whose corresponding open subsets cover* $\mathscr{S}(\mathscr{C})$*.*

(2) *The identity functor* $(\mathrm{Sym}^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Sym}$ *is a sheaf with respect to this Grothendieck topology.*

(3) *The posets of open (resp. closed) immersions also satisfies descent for this Grothendieck topology.*

Remark 5.1.7. There is also a variant of Theorem 5.1.6 where the covers are given by finitely many closed localizations whose closed subspaces cover $\mathscr{S}(\mathscr{C})$. Later we will see that a more general Grothendieck topology, called the !-topology, has open and closed covers of symmetric monoidal categories as particular covers.

5.2. **The ring $\mathbb{Z}[T]_{\square}$.** Let $P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ be the condensed abelian group of null sequences and let $\mathbb{Z}[\hat{q}]$ be P considered as an algebra. The multiplication by $q = [1]$ in $\mathbb{Z}[\hat{q}]$ corresponds to the shift map. Since $\mathbb{Z}[\hat{q}] \subset \mathbb{Z}[[q]]$, it is an integral domain and so we have a short exact sequence

$$0 \to \mathbb{Z}[\hat{q}] \xrightarrow{1-q} \mathbb{Z}[\hat{q}] \to \mathbb{Z}[\hat{q}]/(1-q) \to 0.$$

By definition Solid is the full subcategory of condensed abelian groups M such that

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[\hat{q}]/(1-q), M) = 0.$$

Note that this localization process fits in the general framework of Proposition 4.6.2 (2) since P is internally compact projective in condensed abelian groups. From Theorem 3.2.3 (12) we know that

$$\mathbb{Z}[\hat{q}]^{L\square} = \mathbb{Z}[[q]].$$

Let us consider the polynomial algebra in one variable $\mathbb{Z}[T]$ seen as a solid abelian group. We let $A = (\mathbb{Z}[T], \mathbb{Z})_{\square}$ denote the induced analytic structure $\mathbb{Z}[T]_{\mathbb{Z}_{\square}/}$. Then

$$A \otimes_{\mathbb{Z}} P = (\mathbb{Z}[T], \mathbb{Z})_{\square} \otimes_{\mathbb{Z}} \mathbb{Z}[\hat{q}] = \mathbb{Z}[[q]][T]$$

is a polynomial algebra over the power series ring in the variable q. Then, we could solidify the variable T by asking that a null-sequence (m_n) in an A-module M is "T-summable", i.e. that $\sum_n m_n T^n$ converges (uniquely and functorially) in M. This leads to the following definition

Definition 5.2.1. An object $M \in \mathrm{Mod}(A)$ (resp. in $\mathscr{D}(A)$) is said $\mathbb{Z}[T]$-solid (or T-solid) if the natural map

$$R\underline{\mathrm{Hom}}_A(A \otimes_{\mathbb{Z}} P, M) \xrightarrow{1-T\,\mathrm{Shift}^*} R\underline{\mathrm{Hom}}_A(A \otimes_{\mathbb{Z}} P, M)$$

is an equivalence. We let $\mathrm{Mod}(\mathbb{Z}[T]_{\square})$ (resp. $\mathscr{D}(\mathbb{Z}[T]_{\square})$) be the full subcategory of $\mathbb{Z}[T]$-solid modules.

Since $A \otimes_{\mathbb{Z}} P$ is a compact projective A-algebra, Proposition 4.6.2 (2) shows that $\mathscr{D}(\mathbb{Z}[T]_{\square})$ is essentially an (uncompleted) analytic ring structure on $A^{\triangleright}$, and the only condition to verify is the right t-exactness of the localization. We will do better, and we will actually compute the free solid modules $\mathbb{Z}[T]_{\square}[S]$ for $S \in \mathrm{Prof}^{\mathrm{light}}$.

To begin with, note that the fiber sequence $A \otimes_{\mathbb{Z}} P \xrightarrow{1-T\,\mathrm{Shift}} A \otimes_{\mathbb{Z}} P \to Q$ is actually exact and that it is equivalent to the short exact sequence

$$0 \to \mathbb{Z}[[q]][T] \xrightarrow{1-Tq} \mathbb{Z}[[q]][T] \to \mathbb{Z}((T^{-1})) \to 0 \tag{5.4}$$

where $\mathbb{Z}((T^{-1})) = \mathbb{Z}[[T^{-1}]][T]$ is the algebra of Laurent power series in the variable T^{-1}. In particular, since $\mathbb{Z}[[X]]$ is an idempotent $(\mathbb{Z}[X], \mathbb{Z})_{\square}$-algebra, one deduces that $\mathbb{Z}((T^{-1}))$ is an idempotent $(\mathbb{Z}[T], \mathbb{Z})_{\square}$-algebra. We get the following proposition.

Proposition 5.2.2. *The category $\mathscr{D}(\mathbb{Z}[T]_{\square})$ is the localization of $\mathscr{D}((\mathbb{Z}[T], \mathbb{Z})_{\square})$ with respect to the idempotent algebra $\mathbb{Z}((T^{-1}))$. More precisely, we have a semi-orthogonal decomposition*

$$\mathscr{D}(\mathbb{Z}((T^{-1}))_{\mathbb{Z}_{\square}/}) \xrightarrow{\iota_*} \mathscr{D}((\mathbb{Z}[T], \mathbb{Z})_{\square}) \xrightarrow{j^*} \mathscr{D}(\mathbb{Z}[T]_{\square})$$

where ι_* is the natural inclusion and j^* is the localization functor. We let ι^* be the base change $\mathbb{Z}((T^{-1})) \otimes_A -$ and let j_* be the right adjoint of j^*.

From the general non-sense of smashing localizations in presentably symmetric monoidal stable ∞-categories (see [CS22, Lecture V] and Remark 5.1.5), we can explicitly compute the functor j_*:

$$j_* j^* M = R\underline{\mathrm{Hom}}_A(\mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))), M),$$

in particular the functor $j_* j^*$ commutes with limits.

Using the resolution (5.4) of $\mathbb{Z}((T^{-1}))$ one can easily compute that $R\underline{\mathrm{Hom}}_A(\mathbb{Z}((T^{-1})), \mathbb{Z}[T]) = 0$ so that $j_* j^* \mathbb{Z}[T] = \mathbb{Z}[T]$. Indeed, we can write

$$R\underline{\mathrm{Hom}}_A(\mathbb{Z}[[q]][T], \mathbb{Z}[T]) = \mathbb{Z}[T]((q))/q\mathbb{Z}[T][[q]]$$

as $\mathbb{Z}[q,T]$-module, and multiplication by $1 - Tq$ is invertible on $\mathbb{Z}[T][[q]]$.

With this computation we can show the following theorem (for more details see [CS19, Lecture VIII])

Theorem 5.2.3. *The full subcategory $\mathscr{D}(\mathbb{Z}[T]_{\square}) \subset \mathscr{D}((\mathbb{Z}[T], \mathbb{Z})_{\square})$ defines an analytic ring structure on $\mathbb{Z}[T]$. For $S = \varprojlim_n S_n$ a light profinite set written as limit of finite sets, the free $\mathbb{Z}[T]_{\square}$-module generated by S is given by*

$$\mathbb{Z}[T]_{\square}[S] = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}[T][S_n].$$

Sketch of the proof. Let us write $A = (\mathbb{Z}[T], \mathbb{Z})_{\square}$. Conditions (1) and (2) of Definition 4.1.1 follow immediately from Proposition 4.6.2 and the fact that the object $A \otimes_{\mathbb{Z}} P/(1 - Tq)$ is compact in $\mathscr{D}((\mathbb{Z}[T], \mathbb{Z})_{\square})$. It is left to show condition (3), this one follows from the computation of $\mathbb{Z}[T]_{\square}[S]$. Recall that, if S is infinite, $\mathbb{Z}_{\square}[S] \cong \prod_{\mathbb{N}} \mathbb{Z}$. Since $\mathscr{D}(\mathbb{Z}[T]_{\square})$ is the localization with respect to the objects in $\mathscr{D}(\mathbb{Z}((T^{-1})))$, it suffices to show that the cone Q of the map $(\prod_{\mathbb{N}} \mathbb{Z})[T] \to \prod_{\mathbb{N}} (\mathbb{Z}[T])$ is a $\mathbb{Z}((T^{-1}))$-module. But it is not hard to see that

$$\prod_{\mathbb{N}} (\mathbb{Z}[T])/((\prod_{\mathbb{N}} \mathbb{Z})[T]) = \prod_{\mathbb{N}} (\mathbb{Z}((T^{-1})))/((\prod_{\mathbb{N}} \mathbb{Z}) \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}((T^{-1}))).$$

One deduces that

$$j_* j^* Q = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}(\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))), Q) = 0$$

as Q is a $\mathbb{Z}((T^{-1}))$-module. We get that

$$\begin{aligned}\mathbb{Z}[T]_{\square}[S] &= j_*j^*(\mathbb{Z}_{\square}[S][T]) \\ &= j_*j^*(\varprojlim_n \mathbb{Z}[T][S_n]) \\ &= \varprojlim_n (j_*j^*\mathbb{Z}[T][S_n]) \\ &= \varprojlim_n \mathbb{Z}[T][S_n]\end{aligned}$$

finishing the proof of the theorem. □

5.3. **Solid rings of finite type algebras.** Let us now generalize the construction of the ring $\mathbb{Z}[T]_{\square}$ to finite type algebras over $\mathbb{Z}$.

Definition 5.3.1.

(1) Let R be a solid animated algebra and let $r \in \pi_0(R)$. We say that a solid R-module M is r-solid if for (any and so all) lifts $\mathbb{Z}[T] \to R$ of r, the restriction of M to a $\mathbb{Z}[T]$-module is T-solid.

(2) Let R be a finite type algebra over $\mathbb{Z}$. We let $R_{\square}$ be the analytic ring structure on R making an R-module M complete if and only if M is r-solid for all $r \in R$.

Theorem 5.3.2. *Let R be a finite type algebra over $\mathbb{Z}$. Then for $S = \varprojlim_n S_n$ a light profinite set the natural map*

$$R_{\square}[S] \to \varprojlim_n R[S_n]$$

is an isomorphism. In particular, $\mathscr{D}(R_{\square})$ is the derived category of its heart and $\otimes^L_{R_{\square}}$ is the left derived functor of $\otimes_{R_{\square}}$. Moreover, $\prod_n R$ is flat for the $R_{\square}$-tensor product.

Our first task is to compute the free solid generators $R_{\square}[S]$ for $S \in \mathrm{Prof}^{\mathrm{light}}$. Note that any algebra of finite type is a quotient of a polynomial algebra, let us then start with those:

Proposition 5.3.3. *Let $T_1, \ldots, T_n$ be a set of variables, then the natural map*

$$\mathbb{Z}[T_1]_{\square} \otimes_{\mathbb{Z}_{\square}} \cdots \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[T_n]_{\square} \to \mathbb{Z}[T_1, \ldots, T_n]_{\square} \tag{5.5}$$

is an isomorphism.

Proof. Let $A = \mathbb{Z}[T_1]_{\square} \otimes_{\mathbb{Z}_{\square}} \cdots \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[T_n]_{\square}$, it is the analytic ring structure on $\mathbb{Z}[T_1, \ldots, T_n]$ making a module A-complete if and only it it is $\mathbb{Z}[T_i]_{\square}$-complete for all $i = 1, \ldots, n$. We clearly have that $\mathscr{D}(\mathbb{Z}[T_1, \ldots, T_n]_{\square}) \subset \mathscr{D}(A) \subset \mathscr{D}^{\mathrm{cond}}(\mathbb{Z}[T_1, \ldots, T_n])$. We need to show the opposite inclusion $\mathscr{D}(A) \subset \mathscr{D}(\mathbb{Z}[T_1, \ldots, T_n]_{\square})$, i.e. that if a solid $\mathbb{Z}[T_1, \ldots, T_n]$-module is T_i-solid for all $i + 1, \ldots, n$, then it is $p(T)$-solid for all $p(T) \in \mathbb{Z}[T_1, \ldots, T_n]$.

As a first step, let us compute the compact projective generators of the ring A. We show by induction on the number of variables that for $S = \varprojlim_k S_k$ profinite

$$A[S] = \varprojlim_n \mathbb{Z}[T_1, \ldots, T_n][S_k],$$

the case $n = 1$ being Theorem 5.2.3. Suppose that the claim follows for n and consider $B = \mathbb{Z}[T_1, \ldots, T_n, T_{n+1}]_{A/}$ the induced analytic rings structure. Let $C = \mathbb{Z}[T_1]_{\square} \otimes_{\mathbb{Z}_{\square}} \cdots \otimes_{\mathbb{Z}_{\square}} \mathbb{Z}[T_n]_{\square}$. By definition $\mathscr{D}(C) \subset \mathscr{D}(B)$ is the full subcategory of objects that are $\mathbb{Z}[T_{n+1}]$-solid (since they are already $\mathbb{Z}[T_i]$-solid for all $i = 1, \ldots, n$). By Proposition 5.2.2 an object $M \in \mathscr{D}(A)$ is $\mathbb{Z}[T_{n+1}]$-solid if and only if

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T_{n+1}]}(\mathbb{Z}((T_{n+1}^{-1})), M) = 0.$$

By taking base change along $\mathbb{Z}[T_{n+1}]_{\mathbb{Z}_{\square}} \to B$ this is equivalent to the vanishing of

$$R\underline{\mathrm{Hom}}_{\mathbb{Z}[T_1, \ldots, T_{n+1}]}(B \otimes_{\mathbb{Z}[T_{n+1}]} \mathbb{Z}((T_{n+1}^{-1})), M) = 0.$$

Recall that we have the resolution

$$0 \to \mathbb{Z}[[q]][T_{n+1}] \xrightarrow{1-qT_{n+1}} \mathbb{Z}[[q]][T_{n+1}] \to \mathbb{Z}((T_{n+1}^{-1})) \to 0.$$

Then, by induction, we have that

$$B \otimes_{\mathbb{Z}[T_{n+1}]} \mathbb{Z}((T_{n+1}^{-1})) = \mathbb{Z}[T_1, \ldots, T_n]((T_{n+1}^{-1})),$$

which is an idempotent B-algebra. Then, the same argument of Theorem 5.2.3 will show that

$$C_{\square}[S] = \varprojlim_{n} \mathbb{Z}[T_1, \ldots, T_{n+1}][S_k]$$

as wanted.

Now, note that any discrete $\mathbb{Z}[T_1, \ldots, T_n]$-module is immediately $\mathbb{Z}[T_1, \ldots, T_n]_{\square}$-complete since it is a-solid for any $a \in \mathbb{Z}[T_1, \ldots, T_n]$. This shows that $A[S] \cong \prod_k \mathbb{Z}[T_1, \ldots, T_n]$ is $\mathbb{Z}[T_1, \ldots, T_n]_{\square}$-complete (since complete modules are stable under products), and so that any complete A-module is $\mathbb{Z}[T_1, \ldots, T_n]_{\square}$-complete (being stable under colimits). One deduces that

$$\mathscr{D}(A) \subset \mathscr{D}(\mathbb{Z}[T_1, \ldots, T_n]_{\square}) \subset \mathscr{D}^{\mathrm{cond}}(\mathbb{Z}[T_1, \ldots, T_n])$$

and so that we have the equivalence

$$\mathscr{D}(A) = \mathscr{D}(\mathbb{Z}[T_1, \ldots, T_n]_{\square}),$$

proving that the map (5.5) is indeed an equivalence. □

Corollary 5.3.4. *Let R be a finite type algebra and let $\mathbb{Z}[T_1, \ldots, T_n] \to R$ be a surjection. Then*

$$R_{\square} = R_{\mathbb{Z}[T_1,\ldots,T_n]_{\square}/}$$

has the induced analytic structure. Moreover, for $S = \varprojlim_k S_k$ a light profinite set we have

$$R_{\square}[S] = \varprojlim_{k} R[S_n].$$

Proof. By definition an M-module is $\mathbb{Z}[T_1, \ldots, T_n]_{\square}$-complete if it is $\mathbb{Z}[a]_{\square}$-complete for all $a \in \mathbb{Z}[T_1, \ldots, T_n]$. By definition of $R_{\square}$ this shows that it has the induced analytic structure. In particular,

$$R_{\square}[S] = R \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]_{\square}} \mathbb{Z}[T_1, \ldots, T_n]_{\square}[S].$$

We will prove a more general fact: let M be a finite type $\mathbb{Z}[T_1, \ldots, T_n]$-module, then

$$M \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]_{\square}} \prod_k (\mathbb{Z}[T_1, \ldots, T_n]_{\square}) = \prod_k M.$$

Indeed, consider a finite projective resolution $P_{\bullet} \to M$ where all the terms P_n are finitely many copies of $\mathbb{Z}[T_1, \ldots, T_n]$. Then $M \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]_{\square}} \prod_k (\mathbb{Z}[T_1, \ldots, T_n]_{\square})$ is equivalent to the complex $P_{\bullet} \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]_{\square}} \prod_k (\mathbb{Z}[T_1, \ldots, T_n]_{\square})$. Since each term P_n is a finite free module we actually have

$$P_{\bullet} \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]_{\square}} \prod_k (\mathbb{Z}[T_1, \ldots, T_n]_{\square}) = \prod_k P_{\bullet}.$$

Since countable products are exact we have an equivalence

$$\prod_k P_{\bullet} \xrightarrow{\sim} \prod_k M.$$

□

Proof of Theorem 5.3.2. The claim about the free objects on profinite sets is Corollary 5.3.4. The fact that $\mathscr{D}(R_{\square})$ is the derived category of its heart and that $\otimes^L_{R_{\square}}$ is the left derived functor of $\otimes_{R_{\square}}$ follows the same argument of the analogue statements in Theorem 3.2.3. The final statement about flatness of $\prod_n R$ will be proven in Proposition 5.3.7. □

It is left to prove flatness of $\prod_n R$, this will follow a similar argument as the one for $\mathbb{Z}$.

Definition 5.3.5. An $R_\square$-module is finitely generated if it is a quotient of $\prod_{\mathbb{N}} R$. An $R_\square$-module is of finite presentation (or coherent) if it is a cokernel of a map $\prod_{\mathbb{N}} R \to \prod_{\mathbb{N}} R$.

We want to have a good understanding of coherent $R_\square$-modules. Under some topological hypothesis these are easier to describe:

Lemma 5.3.6. *If $M \in \mathrm{Mod}_{R_\square}$ is quasi-separated, then the following are equivalent.*

i. *M is of finite presentation.*
ii. *M is finitely generated.*
iii. *$M = \varprojlim M_n$ is a limit of finitely generated discrete R-modules with surjective transition maps.*

Proof. Let $(M_n)_n$ be a projective system as in (iii). Since R is noetherian we can find a right exact resolution

$$R^{s_n} \to R^{k_n} \to M_n \to 0$$

for each n. By constructing the resolution step by step, we can construct liftings

$$\begin{array}{ccccccc} R^{s_{n+1}} & \longrightarrow & R^{k_{n+1}} & \longrightarrow & M_{n+1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ R^{s_n} & \longrightarrow & R^{k_n} & \longrightarrow & M_n & \longrightarrow & 0 \end{array}$$

such that all the vertical maps are surjective. Taking limits we find a right exact sequence

$$\prod_{\mathbb{N}} R \to \prod_{\mathbb{N}} R \to M \to 0$$

proving (iii)⇒ (ii).

It is clear that (ii) ⇒ (i). It is left to show that (i) implies (iii). We have a surjection $\prod_n R \to M \to 0$ with kernel K. The space $\prod_n R$ arises from a metrizable topological space, and since M is quasi-separated the space $K \subset \prod_n R$ is closed. Writing $\prod_n R = \varprojlim_n R^n$ as the limit of finite free R-modules, we see that $K = \varprojlim_n K_n$ where K_n is the image of K in the projection $\prod_n R \to R^n$. This shows that

$$M = \varprojlim_n R^n/K_n$$

proving what we wanted. □

Proposition 5.3.7. *Let $\mathrm{Mod}_{R_\square}$ be the abelian category of $R_\square$-modules and let $\mathrm{Mod}^{\mathrm{coh}}_{R_\square} \subset \mathrm{Mod}_{R_\square}$ be the full subcategory of coherent modules. The following hold:*

(1) *We have $\mathrm{Mod}_{R_\square} = \mathrm{Ind}(\mathrm{Mod}^{\mathrm{coh}}_{R_\square})$.*
(2) *The category $\mathrm{Mod}^{\mathrm{coh}}_{R_\square}$ of coherent modules is an abelian category stable under all kernels, cokernels and extensions.*
(3) *Any coherent module M is pseudo-coherent, namely, it has a resolution of the form $P_\bullet \to M$ with $P_\bullet$ a free $R_\square$-module on a profinite set S.*
(4) *The $R_\square$-module $\prod_n R$ is flat.*

Proof. (1) The category $\mathrm{Mod}_{R_\square}$ has a compact projective generator given by $\prod_{\mathbb{N}} R$. This formally shows that coherent modules are the compact objects in $\mathrm{Mod}_{R_\square}$ and the description as inductive category of (1).

(2) By the standard arguments in commutative algebra it suffices to show that any finitely generated module M of $\prod_{\mathbb{N}} R$ is actually finitely presented. By then M is quasi-separated and Lemma 5.3.6 implies that M is coherent.

(3) Let M be a coherent module and consider a right exact sequence

$$\prod_{\mathbb{N}} R \xrightarrow{f} \prod_{\mathbb{N}} R \to M \to 0.$$

By part (2) the kernel $K = \ker f$ is a coherent module. An inductive argument allow us to construct the resolution $P_\bullet \to M$ of M where each term is isomorphic to $\prod_{\mathbb{N}} R$.

(4) It suffices to show that for any coherent module M we have $M \otimes^L_{R_\square} \prod_{\mathbb{N}} R = \prod_{\mathbb{N}} M$ as then the tensor product will be exact. This follows from the fact that $P \otimes^L_{R_\square} \prod_{\mathbb{N}} R = \prod_{\mathbb{N}} P$ for $P \cong \prod_{\mathbb{N}} R$ and the resolution of M of part (3).

□

Remark 5.3.8. Let R be a finite type $\mathbb{Z}$-algebra. Then for any pseudo-coherent solid $R_\square$-module we have $M \otimes^L_{R_\square} \prod_{\mathbb{N}} R = \prod_{\mathbb{N}} M$, namely, this follows from the computation of the solid tensor product of countable products of R, and the fact that M has a projective resolution whose terms are given $\prod_{\mathbb{N}} R$

As a special consequence, if R is an animated algebra with $\pi_0(R)$ a finitely generated $\mathbb{Z}$-algebra and $\pi_i(R)$ a finite $\pi_0(R)$-module for all i, then the induced analytic ring structure $R_\square := R_{\pi_0(R)_\square /}$ from Corollary 4.4.4 is such that

$$R_\square[S] = \varprojlim_n R[S_n],$$

so that $\prod_I R \otimes_{R_\square} \prod_J R = \prod_{I\times J} R$ for countable sets I and J.

The solid ring structures are also independent of integral extensions.

Corollary 5.3.9. *Let $R \to A$ be an integral map of finitely generated algebras. Then the natural map of analytic rings $A_{R_\square /} \to A_\square$ is an isomorphism.*

Proof. By hypothesis both R and A are finitely generated $\mathbb{Z}$-algebras with A integral over R, so that A is a finite R-module. By Remark 5.3.8 we find that $A \otimes_{R_\square} \prod_{\mathbb{N}} R = \prod_{\mathbb{N}} A$, which implies the corollary. □

5.4. **Schemes as analytic stacks.** With the introduction of the rings $R_\square$ for R a finitely generated $\mathbb{Z}$-algebra we are in shape to talk about two different realizations of schemes as analytic stacks (these will be introduced later in the notes).

5.4.1. *Classical approach.* We first need to see commutative rings as analytic rings:

Proposition 5.4.1. *Let* AniRing *be the ∞-category of (discrete) animated rings. There is a fully faithful embedding*

$$\underline{(-)} : \mathrm{AniRing} \to \mathrm{AnRing}$$

mapping a ring R to the analytic ring $\underline{R} = (R, \mathscr{D}(\underline{R}))$ to the trivial analytic ring structure on R, i.e. the analytic ring structure whose complete modules are all condensed R-modules.

Proof. This follows from the natural fully faithful embedding of animated rings into animated condensed rings as discrete rings. □

Let R be an animated commutative ring and let $\mathrm{Spec}\, R$ be its spectrum defined as the spectrum of $\pi_0(R)$. Let $\mathscr{D}(R)$ be the ∞-derived category of R-modules. The Zariski topology of $\mathrm{Spec}\, R$ has a basis of open affine schemes given by spectrums of the form $\mathrm{Spec}\, R[f^{-1}]$ for $f \in R$ (were by definition $R[f^{-1}] = \varinjlim_{\times f} R$ is the colimit of multiplication by f). We can rephrase the classical Zariski descent of R-modules in the language of Theorem 5.1.6 and Remark 5.1.7.

Proposition 5.4.2. *Let $\{U_i = \mathrm{Spec}\, R_i\}_{i=1}^n$ be a finite affine Zariski cover of* $\mathrm{Spec}\, R$*. Then the morphisms of symmetric monoidal categories $\{f_i^* : \mathscr{D}(R) \to \mathscr{D}(R_i)\}_{i=1}^n$ form a closed cover of $\mathscr{D}(R)$. In particular, we have descent of quasi-coherent sheaves on* $\mathrm{Spec}\, R$ *for the Zariski topology.*

Proof. We prove the proposition in two steps.

Step 1. Let $f_1, \ldots, f_n \in R$ be elements generating the unit ideal and suppose that the cover is of the form $U_i = \operatorname{Spec} R[f_i^{-1}]$. It is clear that the R-algebras $R[f_i^{-1}]$ are idempotent so that they define closed subspaces in the locale $\mathscr{S}(\mathscr{D}(R))$. We want to see that they cover the locale, but this is equivalent to asking that there is an equivalence of complexes

$$R \xrightarrow{\sim} [\bigoplus_{i=1}^{n} R[\frac{1}{f_i}] \to \bigoplus_{i<j} R[\frac{1}{f_i f_j}] \to \cdots \to R[\frac{1}{f_1 \cdots f_n}]]$$

which amounts to Zariski descent for the underlying ring.

Step 2. Now let $\{U_i\}_{i=1}^n$ be an arbitrary open cover of R. By Zariski descent we know that there is a natural equivalence of complexes

$$R \xrightarrow{\sim} [\bigoplus_{i=1}^{n} R_i \to \bigoplus_{i<j} R_i \otimes_R R_j \to \cdots \to \bigotimes_{i,R} R_i]$$

so the only thing to show is that each R-algebra R_i is idempotent. But for any open affine subspace $U = \operatorname{Spec} R'$ of R there is a Zariski cover of U of the form $R[f_i^{-1}]$ for suitable $f_i \in R$. It is clear that $R[f_i^{-1}] = R'[f_i^{-1}]$ so that we have an equivalence

$$R' \xrightarrow{\sim} [\bigoplus_{i=1}^{n} R[\frac{1}{f_i}] \to \bigoplus_{i<j} R[\frac{1}{f_i f_j}] \to \cdots \to R[\frac{1}{f_1 \cdots f_n}]],$$

but the right hand side is the idempotent algebra in $\mathscr{D}(R)$ corresponding to the union of the closed subspaces of the locale $\mathscr{S}(\mathscr{D}(R))$ associated to the algebras $R[f_i^{-1}]$. This shows that R' is idempotent which finishes the proof. □

An immediate corollary of Proposition 5.4.2 is the construction of quasi-coherent sheaves for schemes.

Corollary 5.4.3. *Let X be a scheme with structural sheaf $\mathscr{O}_X$ and let $|X|^{\mathrm{op}}$ be the topological space with underlying set $|X|$ and the coarsest topology given by declaring closed subspaces the subsets of the form $|U| \subset X$ with U an open Zariski subspace. Then the functor that maps an open affine Zariski subspace $U \subset X$ to $\mathscr{D}(\mathscr{O}_X(U))$ is a sheaf. More precisely, let $\mathscr{D}(X) = \varprojlim_{U \subset X} \mathscr{D}(\mathscr{O}_X(U))$ be the category of quasi-coherent sheaves on X, where U runs over the poset of open affine subspaces of X. Then there is a unique natural morphism of locales*

$$F : \mathscr{F}(\mathscr{D}(X)) \to |X|^{\mathrm{op}}$$

such that for $U \subset X$ open affinoid we have $\mathscr{D}(F^{-1}(U)) = \mathscr{D}(\mathscr{O}_X(U))$.

Proof. This is a consequence of Proposition 5.4.2, the only thing to verify is that the map is surjective. For the last claim, let $U \subset X$ be a Zariski open subset and let A_U be the idempotent algebra associated to U. Then we have that

$$U = \{x \in X : k(x) \otimes_{\mathscr{O}_X} A_U \neq 0\}.$$

This shows that if for two Zariski open sets one has $F^{-1}(U) = F^{-1}(U')$, i.e. $A_U = A_{U'}$, then $U = U'$ proving surjectivity. □

In Corollary 5.4.3 it is not relevant that we have used the classical category of quasi-coherent sheaves. Indeed, the same holds if we see $R = \underline{R}$ as a discrete condensed ring and we take $\mathscr{D}(\underline{R})$ to be the category of condensed R-modules: the only feature that is needed are the idempotent properties of the algebras $R[f^{-1}]$ for $f \in R$ and the classical Zariski descent for the underlying rings. [4]

However, an anti-intuitive phenomena is happening in the classical Zariski descent of quasi-coherent sheaves, namely, open Zariski subspaces of $\operatorname{Spec} R$ are giving rise to closed subspaces of the locale $\mathscr{S}(\mathscr{D}(R))$! This explains why the classical theory of quasi-coherent sheaves on schemes do not have (apparently!) a well defined theory of cohomology with compact support outside the proper case. A way to overcome this discrepancy is to use a different realization of schemes into analytic stacks, or equivalently, a different embedding of the category of commutative rings into the category of analytic rings. A the end, it will be more convenient to work with a generalization of both theories of quasi-coherent sheaves; this is captured in the theory of discrete adic spaces (see [CS19, Lectures IX and X]). But before that, let us finish the study of the topology of the locale defined by classical algebraic geometry.

5.4.2. *Formal completions.* As we saw above for $U = \operatorname{Spec} R' \to \operatorname{Spec} R$ an open Zariski map gives rise to a closed subspace of the locale $\mathscr{S}(\mathscr{D}(R))$. A natural question is to describe what the open complement is, it has indeed a very natural answer.

Proposition 5.4.4. *Let X be a scheme and let $U \subset X$ be a qcqs open Zariski subspace with closed complement $Z \subset X$. Let $X^{\wedge Z}$ be the (derived) formal completion of X along Z. Let $\mathscr{D}(X)$ be the ∞-category of derived quasi-coherent sheaves on X and let $F : \mathscr{S}(\mathscr{D}(X)) \to |X|^{\mathrm{op}}$ be the map of locales provided by Corollary 5.4.3. Then the open subspace $F^{-1}(Z)$ has as underlying category the derived category of (derived) formally complete quasi-coherent sheaves on $X^{\wedge Z}$.*

Sketch of the proof. We will explain the simplest case when $X = \operatorname{Spec} R$ is affine and $U = \operatorname{Spec} R[f^{-1}]$ is given by inverting $f \in R$. For a general definition of I-adically complete modules we refer to [Man22, Definition 2.12.3], and we left as an exercise to the reader the formal definition of $X^{\wedge Z}$ and $\mathscr{D}(X^{\wedge Z})$, and the proof of the proposition in the general case (Hint: $X^{\wedge Z}$ is actually an ind-scheme).

By definition $\mathscr{D}(F^{-1}(Z))$ is the Verdier quotient $j^* : \mathscr{D}(R) \to \mathscr{D}(R)/\mathscr{D}(R[f^{-1}])$. Moreover, the functor $j_* : \mathscr{D}(F^{-1}(Z)) \to \mathscr{D}(R)$ is fully faithful and as by essential image the elements $M \in \mathscr{D}(R)$ such that the natural map

$$M \to R\mathrm{Hom}_R(\mathrm{fib}(R \to R[f^{-1}]), M)$$

is an equivalence (equivalently those M such that $R\mathrm{Hom}_R(R[f^{-1}], M) = 0$). But we can write $R[f^{-1}] = \varinjlim_{\times f} R$ as the colimit of multiplication by f on R, and so we find that

$$R\mathrm{Hom}_R(\mathrm{fib}(R \to R[f^{-1}]), M) = R\varprojlim_n (M \otimes_R^L R/^{\mathbb{L}} f^n)$$

where $R/^{\mathbb{L}} f^n$ is the derived quotient of R by f^n, represented by the Koszul complex $[R \xrightarrow{f^n} R]$. Thus, by definition, the essential image of f consists on those modules which are derived f-adically complete.

Now let $I = (f) \subset R$ be the ideal generated by f. To finish the proof of the proposition it suffices to see that for any element $g \in \mathrm{Rad}(I)$ in the radical of I, a derived f-adically complete module is also derived g-adically complete. Indeed, this will show that the category $\mathscr{D}(F^{-1}(Z))$ only depends on the formal completion of $\operatorname{Spec} R$ along Z and that by definition it consists on the derived formally complete modules of $X^{\wedge Z}$. To see this, we note that $\operatorname{Spec} R[g^{-1}] \subset \operatorname{Spec} R[f^{-1}]$,

[4]Zariski descent can go even more general in the following way: let $\mathrm{Pr}^L_{\mathscr{D}(R)}$ be the $(\infty, 2)$-category of $\mathscr{D}(R)$-linear presentable categories. Given a morphism of rings $R \to S$ there is a natural base change functor $\mathrm{Pr}^L_{\mathscr{D}(R)} \to \mathrm{Pr}^L_{\mathscr{D}(S)}$ given by Lurie's tensor product $M \mapsto M \otimes_{\mathscr{D}(R)} \mathscr{D}(S)$. Then the functor mapping an open affine Zariski subspace $U = \operatorname{Spec} R' \subset \operatorname{Spec} R$ to $\mathrm{Pr}^L_{\mathscr{D}(R')}$ is a sheaf for the Zariski topology.

i.e. that we have a map of idempotent R-algebras $R[f^{-1}] \to R[g^{-1}]$. But then

$$\begin{aligned} R\mathrm{Hom}_R(R[g^{-1}], M) &= R\mathrm{Hom}_R(R[g^{-1}] \otimes_R^L R[f^{-1}], M) \\ &= R\mathrm{Hom}_R(R[g^{-1}], R\mathrm{Hom}_R(R[f^{-1}], M)) \\ &= 0 \end{aligned}$$

proving what we wanted. □

To conclude, classical algebraic geometry is in the strange situation where the classical qcqs open Zariski maps give rise to closed maps at the level of derived categories, while formal completions along finitely generated Zariski closed subschemes give rise to open immersions.

5.4.3. *Solid approach.* Let us now discuss the solid approach to the theory of quasi-coherent sheaves for schemes. We need a definition:

Definition 5.4.5. Let R be a commutative ring, we define the analytic ring $R_{\square}$ to be the colimit

$$R_{\square} = \varinjlim_{B \subset R} B_{\square}$$

where B runs over all the finitely generated $\mathbb{Z}$-subalgebras, and $B_{\square}$ is the analytic ring constructed in Theorem 5.3.2. Equivalently, $B_{\square}$ is the analytic ring structure on B where a condensed B-module is $B_{\square}$-complete if and only if it is b-solid for all $b \in B$.

In general, for R an animated commutative ring we let $R_{\square}$ be the analytic ring structure on R induced from $\pi_0(R)_{\square}$ via Proposition 4.4.4. Equivalently, an R-module M is $R_{\square}$-complete if and only if all its cohomology groups $H^*(M)$ are $\pi_0(R)_{\square}$-complete.

Remark 5.4.6. By Theorem 5.3.2, for B a finitely generated $\mathbb{Z}$-algebra $B_{\square}$ is an analytic ring structure of B, i.e. B is $B_{\square}$-complete. It follows formally that any discrete (derived) B-module is also $B_{\square}$-complete. This shows that, for any animated ring R, the ring itself is $R_{\square}$-complete and so $R_{\square}$ is an analytic ring structure on R.

Remark 5.4.7. Let R be an animated ring and let $S \subset \pi_0(R)$ be a set of generators as $\mathbb{Z}$-algebra. By Corollary 5.3.4 the ring $R_{\square}$ is also obtained as the analytic ring structure on R making an R-module complete if and only if it is s-solid for all $s \in S$.

Remark 5.4.8. The reader might ask why not to define $B_{\square}$ to be the analytic ring whose values at a profinite set $S = \varprojlim_n S_n$ are given by $B_{\square}[S] = \varprojlim_n B[S_n]$. The answer is that it is not clear (and probably false) that for an arbitrary ring this definition gives rise to an analytic ring structure on B. [5] So far we have seen that this is a sensitive definition when B is a finitely generated $\mathbb{Z}$-algebra. It turns out that this also works well when B is *essenitally of finite type*, i.e. when it is a Zariski localization (i.e. colimit of open Zariski localizations) of a finitely generated $\mathbb{Z}$-algebra. For example, when $B = \mathbb{Q}$ one obtains the *ultra solid* rational numbers and the category $\mathscr{D}(\mathbb{Q}_{\square}) \subset \mathscr{D}(\mathbb{Z}_{\square})$ is the open localization which is complement to the idempotent $\mathbb{Z}_{\square}$-algebra $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.

In general, by going beyond the notion of analytic ring, one can define categories of ultra-solid modules over arbitrary animated rings and $\mathbb{E}_{\infty}$-algebras, see [MA24] for the proper definition and the development of this theory.

Let $R \to S$ be a map of animated commutative rings, by definition of the analytic ring structure there is a map of analytic rings $R_{\square} \to S_{\square}$. Indeed, a complete $S_{\square}$-module is an S-module M which is solid for all $s \in S$, this means concretely that for all map $\mathbb{Z}[T] \to R$ the module M is $\mathbb{Z}[T]_{\square}$-solid. Hence, it is easy to see that its restriction to an R-module is also $R_{\square}$-complete. We get a functor

$$(-)_{\square} : \mathrm{AniRing} \to \mathrm{AnRing}$$

[5]The problem being that in the correct definition of the (ultra-)solid modules attached to B, the forgetful functor $\mathscr{D}(B_{\square}) \to \mathscr{D}(\underline{B})$ from ultra-solid to condensed B-modules could not be fully faithful.

from animated commutative rings to analytic rings.

One of the first properties we have to check is the compatibility with colimits of the functor $(-)_{\square}$.

Proposition 5.4.9. *The functor* $(-)_{\square} : \mathrm{AniRing} \to \mathrm{AnRing}$ *is fully faithful and commutes with colimits.*

Proof. Let R, S be animated rings. We want to see that the natural fully-faithful map of anima

$$(-)^{\triangleright} : \mathrm{Map}_{\mathrm{AnRing}}(R_{\square}, S_{\square}) \to \mathrm{Map}_{\mathrm{AniRing}}(R, S)$$

is essentially surjective. For this, recall that we have a map $\mathrm{AnRing} \to \mathrm{Cond}(\mathrm{AniRing})$ mapping an analytic ring to its underlying condensed ring. The claim follows from the fact that the composite

$$\mathrm{AniRing} \xrightarrow{(-)_{\square}} \mathrm{AnRing} \xrightarrow{(-)^{\triangleright}} \mathrm{Cond}(\mathrm{AniRing})$$

is fully faithful.

Next, we show the second statement. Let $\{R_i\}_{i\in I}$ be a diagram of animated commutative rings and let $\{R_{i,\square}\}_{i\in I}$ be its associated diagram of solid rings. By Proposition 4.1.10 the colimit $\varinjlim_i R_{i,\square}$ is the analytic ring given by the completion of the analytic ring structure on $S := \varinjlim_i R_i$ where an S-module M is complete if and only if it is $R_{i,\square}$-complete for all i. But this is equivalent for M to be r-solid for all $r \in R_i$ and all $i \in I$, which is also equivalent to be s-solid for all $s \in S$. This shows that $\varinjlim_i R_{i,\square} = S_{\square}$ as wanted. $\square$

Our next step to construct solid analytic stacks attached to schemes is proving Zariski descent for the functor mapping $\mathrm{Spec}\, R \mapsto \mathscr{D}(R_{\square})$. We first prove this for coverings given by basic open Zariski subspaces. We need some technical lemmas.

Lemma 5.4.10. *Let $f : A \to B$ be a map of analytic rings such that induced map of derived categories $f^* : \mathscr{D}(A) \to \mathscr{D}(B)$ is an open immersion of locales. Then for all analytic ring C and any map $A \to C$ the base change $C \to C \otimes_A B$ also induces an open immersion at the level of locales.*

Proof. By hypothesis there is $D \in \mathscr{D}(A)$ an idempotent algebra such that $\mathscr{D}(B) \subset \mathscr{D}(A)$ is the full subcategory consisting on those M such that $R\underline{\mathrm{Hom}}_A(D, M) = 0$. This defines an uncompleted analytic ring structure on $A^{\triangleright}$. Thus, by Theorem 4.1.12, $C \otimes_A B$ is given by the completion of the analytic ring structure on $C^{\triangleright}$ such that a $C^{\triangleright}$-module M is complete if and only if it is C and B-complete. This is equivalent to asking M to be C-complete and that $R\underline{\mathrm{Hom}}_C(C \otimes_A D, M) = 0$. Thus, $C \otimes_A B$ is the open localization complement to the idempotent C-algebra $C \otimes_A D$ proving what we wanted. $\square$

Lemma 5.4.11. *The following maps of analytic rings give rise to open immersions at the level of locales:*

(1) $\mathbb{Z}[T]_{\mathbb{Z}_{\square}/} \to \mathbb{Z}[T]_{\square}$ *with complement idempotent algebra* $\mathbb{Z}((T^{-1}))$.
(2) $\mathbb{Z}[T]_{\mathbb{Z}_{\square}/} \to \mathbb{Z}[T^{\pm 1}]_{\mathbb{Z}[T^{-1}]_{\square}/}$ *with complement idempotent algebra* $\mathbb{Z}[[T]]$.

Proof. Part (1) is Theorem 5.2.3. Part (2) follows by a similar argument as in *loc. cit.*, we left the details to the reader. $\square$

Lemma 5.4.12. *Let R be an animated commutative ring and let $f \in R$. Then the map $R_{\square} \to R[f^{-1}]_{\square}$ of analytic rings induces an open immersion at the level of locales.*

Proof. By Proposition 5.4.9 we have that

$$R[f^{-1}]_{\square} = R_{\square} \otimes_{\mathbb{Z}[T]_{\square}} \mathbb{Z}[T^{\pm 1}]_{\square}$$

where $T \mapsto f$. Therefore, by Lemma 5.4.10, it suffices to show that the map $\mathbb{Z}[T]_{\square} \to \mathbb{Z}[T^{\pm 1}]_{\square}$ induces an open immersion of locales, but this follows from Lemma 5.4.11 (2) by taking the base change $\mathbb{Z}[T]_{\mathbb{Z}_{\square}/} \to \mathbb{Z}[T]_{\square}$. $\square$

Proposition 5.4.13. *Let R be an animated commutative ring and let $\{U_i = \operatorname{Spec} R[f_i^{-1}]\}_{i=1}^n$ be an open cover of $\operatorname{Spec} R$ by basic affine subspaces. Then the maps $R_\square \to R[f_i^{-1}]_\square$ induce an open covering in the locale $\mathscr{S}(\mathscr{D}(R_\square))$.*

Proof. By Lemma 5.4.12 we know that the pullback maps $\mathscr{D}(R_\square) \to \mathscr{D}(R[f_i^{-1}]_\square)$ are open localizations. By noetherian approximation we can assume without loss of generality that R is an animated algebra of finite presentation (or for simplicity just a finitely generated algebra over $\mathbb{Z}$). In such a case, the free solid R-module for $S = \varprojlim_n S_n$ profinite is

$$R_\square[S] = \varprojlim_n R[S_n].$$

By construction, the closed complement Z_i of $R_\square \to R[f_i^{-1}]_\square$ is given by the idempotent $R_\square$-algebra

$$R[[f_i]] := R_\square \otimes_{\mathbb{Z}[T]_{\mathbb{Z}_\square/}} \mathbb{Z}[[T]]$$

also given by the derived quotient $R[[X]]/^{\mathbb{L}}(X - f_i)$. Thus, in order to show the proposition it suffices to prove that $\cap_{i=1}^n Z_i = \emptyset$ in $\mathscr{S}(\mathscr{D}(R_\square))$, namely, that $\bigotimes_{i=1}^n R[[f_i]] = 0$. But we have that

$$\bigotimes_{i=1}^n R[[f_i]] = R[[X_1, \ldots, X_n]]/^{\mathbb{L}}(X - f_1, \ldots, X - f_n).$$

Now let us write $1 = \sum_i a_i f_i$ and consider the map $g : \mathbb{Z}[T] \to R[[X_1, \ldots, X_n]]$ mapping T to $\sum_i a_i X_i$. Then g extends (uniquely by idempotency) to a map $\mathbb{Z}[[T]] \to R[[X_1, \ldots, X_n]]$. This shows that the unit $1 \in \bigotimes_{i=1}^n R[[f_i]]$ gives rise to a map from $\mathbb{Z}[[T]]$, but $\mathbb{Z}[[T]]/(T-1) = 0$ proving that $1 = 0$ in the tensor and so that $\bigotimes_{i=1}^n R[[f_i]] = 0$ as wanted. □

Theorem 5.4.14. *Let R be an animated ring and let $\{U_i = \operatorname{Spec} R_i\}_{i=1}^n$ be an open affine Zariski cover of $\operatorname{Spec} R$. Then the maps $R_\square \to R_{i,\square}$ induce an open cover of the locale $\mathscr{S}(\mathscr{D}(R_\square))$.*

Proof. First we show that the maps $R_\square \to R_{i,\square}$ induce open immersion at the level of the locales. For this, let us take open basic affines $\operatorname{Spec} R[f_{i,j}]^{-1}$ covering U_i. We have that $R[f_{i,j}^{-1}] = R_i[f_{i,j}^{-1}]$ and by Proposition 5.4.13 we know that $\{R_{i,\square} \to R_i[f_{i,j}^{-1}]_\square\}_j$ gives rise an open cover of locales. This implies that $\mathscr{D}(R_\square) \to \mathscr{D}(R_{i,\square})$ is an open immersion (being the union of the open immersions $\mathscr{D}(R_\square) \to \mathscr{D}(R_i[f_{i,j}^{-1}]_\square))$. Finally the fact that the open immersions $\{\mathscr{D}(R_\square \to \mathscr{D}(R_{i,\square}))\}_i$ cover $\mathscr{S}(\mathscr{D}(R_\square))$ follows from Proposition 5.4.13 and the fact that we can find a refinement of $\{U_i\}_{i=1}^n$ by basic Zariski open subspaces. □

Corollary 5.4.15. *Let X be a scheme with structural sheaf $\mathscr{O}_X$. Then the functor that maps an open affine Zariski subspace $U \subset X$ to $\mathscr{D}(\mathscr{O}_X(U)_\square)$ is a sheaf. More precisely, let $\mathscr{D}_\square(X) = \varprojlim_{U \subset X} \mathscr{D}(\mathscr{O}_X(U)_\square)$ be the category of solid quasi-coherent sheaves on X, where U runs over the poset of open affine subspaces of X. Then there is a unique natural surjective morphism of locales*

$$F : \mathscr{S}(\mathscr{D}_\square(X)) \to |X|$$

such that for $U \subset X$ open affinoid we have $\mathscr{D}_\square(F^{-1}(U)) = \mathscr{D}(\mathscr{O}_X(U)_\square)$.

Proof. This is a consequence of Theorem 5.4.14, the only thing left to verify is the surjectivity of F. For this, we can assume that $X = \operatorname{Spec} R$ is affine. Let $U, U' \subset X$ be two open Zariski subsets such that $F^{-1}(U) = F^{-1}(U')$, we want to show that $U' = U$. By taking intersections with open affines, we can assume without loss of generality that $U' \subset U = X$. Then, to prove the claim it suffices to show that if for $Z \subset X$ a closed subspace the associated idempotent algebra A_Z is zero, then $Z = \emptyset$. Suppose this does not hold, then $A_Z = 0$ and $Z \neq \emptyset$. There is a closed point $x \in Z$

and a map of analytic rings $R_{\square} \to \kappa(x)_{\square}$ that gives rise a commutative map of locales

$$\begin{array}{ccc} \mathscr{S}(\mathscr{D}(\kappa(x)_{\square})) & \xrightarrow{F_x} & \operatorname{Spec}(\kappa(x)) \\ \downarrow{\scriptstyle G} & & \downarrow \\ \mathscr{S}(\mathscr{D}(R_{\square})) & \xrightarrow{F} & \operatorname{Spec}(R). \end{array}$$

But then we have that $G^{-1}(F^{-1}(Z)) = F_x^{-1}(\operatorname{Spec}(\kappa(x)))$ which shows that $F^{-1}(Z) \neq \emptyset$, a contradiction with the fact that $A_Z = 0$. □

5.5. **Discrete Huber pairs.** In Section 5.4 we saw two possible ways to attach categories of quasi-coherent sheaves to a scheme X (later in the notes we will see that this corresponds to realize the scheme X in two different ways as an analytic stack). For the continuation of the theory it will be more convenient to generalize both constructions in the theory of discrete Huber rings and discrete adic spaces.

Definition 5.5.1. An animated discrete Huber pair is a tuple (R, R^+) where R is an animated discrete ring and $R^+ \subset \pi_0(R)$ is an integrally closed subring. A morphism of discrete Huber pairs $(R, R^+) \to (A, A^+)$ is a map of animated rings $R \to A$ such that R^+ is mapped to A^+. We let AffDis be the ∞-category of animated discrete Huber pairs.

Lemma 5.5.2. *The category* AffDis *admits colimits and has by generators the discrete pairs* $(\mathbb{Z}[T], \mathbb{Z})$ *and* $(\mathbb{Z}[T], \mathbb{Z}[T])$.

Proof. Let $\{(R_i, R_i^+)_{i\in I}$ be a diagram in AffDis, then its colimit is given by the Huber pair (A, A^+) where $A = \varinjlim_i R_i$ and $A^+ \subset \pi_0(A)$ is the integral closure of the ring generated by the images of $R_i^+ \to \pi_0(A)$. To show that $(\mathbb{Z}[T], \mathbb{Z})$ and $(\mathbb{Z}[T], \mathbb{Z}[T])$ generate AffDis, it suffices to show that the functor

$$F : \mathrm{AffDis} \to \mathrm{Ani} \times \mathrm{Ani}$$

mapping a ring (R, R^+) to the mapping spaces $\mathrm{Map}_{\mathrm{AffDis}}((\mathbb{Z}[T], \mathbb{Z}), (R, R^+)) \times \mathrm{Map}_{\mathrm{AffDis}}((\mathbb{Z}[T], \mathbb{Z}[T]), (R, R^+))$ is conservative. But we have that

$$\mathrm{Map}_{\mathrm{AffDis}}((\mathbb{Z}[T], \mathbb{Z}), (R, R^+)) = R$$

and

$$\mathrm{Map}_{\mathrm{AffDis}}((\mathbb{Z}[T], \mathbb{Z}[T]), (R, R^+)) = R \times_{\pi_0(R)} R^+,$$

proving the claim. □

To discrete Huber pairs we can naturally attach a solid analytic ring; it will be convenient to make a more general construction.

Definition 5.5.3. Let (A, S) be a pair consisting on a solid animated ring and a map of sets $S \to \pi_0(A)(*)$ towards the underlying discrete static ring of A. We define the analytic ring $(A, S)_{\square}$ to be completion of the analytic ring structure on A making a condensed A-module complete if and only if it is a solid abelian group and for all $s \in S$ it is s-solid as in Definition 5.2.1.

Lemma 5.5.4. *Let* (R, S) *be a tuple with* R *a discrete animated ring and* $S \subset \pi_0(R)$ *a subset. Let* $R^+[S] \subset \pi_0(R)$ *be the integral closure of the subalgebra generated by the image of* S*. Then the natural map*

$$(R, S)_{\square} \to (R, R^+[S])_{\square}$$

is an equivalence.

Proof. By Proposition 4.4.4 we can assume without loss of generality that R is static. Let (R,S) be a pair with R a discrete ring and $S\subset R$ a set. Let $R^+[S]$ be the integral closure of the subalgebra of R generated by S. We claim that $(R,S)_\square=(R,R^+(S))_\square$. Indeed, let $r\in R^+[S]$, then r is integral over a subalgebra B generated by finitely many elements in S. Then, by Corollary 5.3.9, we have maps of analytic rings

$$B[r]_\square = B[r]_{B_\square/}\to (R,S)_\square,$$

which in turn produces a map of analytic rings

$$(R,B[r])_\square\to (R,S)_\square.$$

Taking colimits along all $r\in R^+[S]$ we get maps $(R,S)_\square\to (R,R^+[S])_\square\to (R,S)_\square$ proving that $(R,R^+[S])_\square=(R,S)_\square$, proving what we wanted. $\square$

Proposition 5.5.5 ([And21, Proposition 3.34]). *The functor* AffDis $\to$ AnRing *mapping* (R,R^+) *to* $(R,R^+)_\square$ *is colimit preserving and fully faithful.*

Proof. First, note that for any pair (R,S) with R a discrete animated ring, the underlying ring $(R,S)_\square^\triangleright$ as any discrete module is solid (resp. s-solid for $s\in S$). The commutativity with colimits follows from the definition of $(R,S)_\square$ by declaring an R-module complete if it is solid as abelian group and s-solid for all $s\in S$, and the description of colimits of (uncompleted) analytic rings of Proposition 4.1.10.

We now prove fully faithfulness. Since the functor is colimit preserving and $(\mathbb{Z}[T],\mathbb{Z})$ and $(\mathbb{Z}[T],\mathbb{Z}[T])$ generate AffDis, it suffices to show that for (R,R^+) a discrete Huber pair the maps

$$R\to \mathrm{Map}_{\mathrm{AnRing}}((\mathbb{Z}[T],\mathbb{Z})_\square,(R,R^+)_\square) \tag{5.6}$$

and

$$R\times_{\pi_0(R)}R^+\to \mathrm{Map}_{\mathrm{AnRing}}((\mathbb{Z}[T],\mathbb{Z}[T])_\square,(R,R^+)_\square) \tag{5.7}$$

are equivalences. By definition, $(\mathbb{Z}[T],\mathbb{Z})_\square=\mathbb{Z}[T]_{\mathbb{Z}_\square/}$ has the induced solid structure from the integers. Therefore, the mapping space (5.6) is the underlying discrete ring of $(R,R^+)_\square$ which is nothing but R as expected. For the second claim, note that the map $(\mathbb{Z}[T],\mathbb{Z})_\square\to(\mathbb{Z}[T],\mathbb{Z}[T])_\square$ is idempotent and so the mapping space of (5.7) is completely determined by its connected components. But by definition $\pi_0(\mathrm{Map}_{\mathrm{AnRing}}((\mathbb{Z}[T],\mathbb{Z}[T])_\square,(R,R^+)_\square))$ consists on all the maps $\mathbb{Z}[T]_\square\to(R,R^+)_\square$. By Proposition 4.4.4 we can assume without loss of generality that R is static, we then have by definition that $(R,R^+)_\square=R_{R^+_\square/}$. Then, for $S=\varprojlim_n S_n$ a light profinite set, we have

$$(R,R^+)_\square[S]=R\otimes_{R^+}R^+_\square[S].$$

Thus, we can write

$$(R,R^+)_\square[S]=\varinjlim_{\substack{B\subset R^+\\ M\subset R}} M_\square[S]$$

where B runs over all the finitely generated subalgebras of R^+, M runs along all the finite B-submodules in R^+, and $M_\square[S]=\varinjlim_n M[S_n]$. Now let $T\in R$ be such that we have a map $\mathbb{Z}[T]_\square\to(R,R^+)_\square$ and let $C\subset R^+$ be the subalgebra generated by the image of T. Then $(R,R^+)_\square[S]$ is $C_\square$-complete and we have a map of $C_\square$-modules

$$C_\square[S]\to(R,R^+)_\square[S]$$

Taking $S=\mathbb{N}\cup\{\infty\}$ and identifying $\mathbb{Z}_\square[S]\cong\prod_{\mathbb{N}}\mathbb{Z}$ we find a map

$$\prod_{\mathbb{N}}C\to\varinjlim_{\substack{B\subset R^+\\ M\subset R}}\prod_{\mathbb{N}}M.$$

In particular, there is a finitely generated ring $B\subset R^+$ and a finite B-module $M\subset R$ such that the sequence $(1,T,T^2,\ldots)$ lands in $\prod_{\mathbb{N}}M$. Thus, all the powers of T are in M which implies that T is integral over B and so that $T\in R^+$. This finishes the proof. $\square$

5.6. **Discrete Adic spaces.** Huber's approximation to non-archimedean geometry relies in the formalism of adic spaces [Hub94, Hub93, Hub96]. The traditional approximation to the theory always involve talking about topological rings and continuous valuations, then some technical sheafy property appears in order to have descent for the analytic topology. In this section we shall follow the approach of [CS19, Lexture IX] and restrict to the theory of discrete adic spaces where these topological and sheafiness subtleties disappear.

5.6.1. *Adic spectrum of Huber pairs.*

Definition 5.6.1. Let (R, R^+) be a discrete Huber pair with R a static ring. We let $\mathrm{Spa}(R, R^+)$ be the set of equivalence classes of multiplicative valuations $x : R \to \Gamma$ such that $|f(x)| \leq 1$ for all $f \in R^+$. For a general discrete Huber pair (R, R^+) we define $\mathrm{Spa}(R, R^+) := \mathrm{Spa}(\pi_0(R), R^+)$. We call $\mathrm{Spa}(R, R^+)$ the *adic spectrum* of (R, R^+). when R^+ is the integral closure of the image of $\mathbb{Z}$ we simply write $\mathrm{Spv}(R)$; this is the set of equivalent classes of valuations of R^+ and is called the valuation spectrum of R. We have $\mathrm{Spa}(R, R^+) \subset \mathrm{Spv}(R)$.

Remark 5.6.2. An equivalent way to define $\mathrm{Spa}(R, R^+)$ is as follows: it is the set of tuples $(\mathfrak{p}, V)$ where $\mathfrak{p}$ is a prime ideal of R and $V \subset \kappa(\mathfrak{p})$ is a valuation ring on the fraction field at $\mathfrak{p}$ containing the image of $R^+ \to \kappa(\mathfrak{p})$.

Huber has defined a topology on $\mathrm{Spa}(R, R^+)$ called the *analytic topology.* It is defined as follows: let (R, R^+) be a discrete Huber pair and let $(f_1, \ldots, f_n, g)$ be a tuple of elements in R. Define the rational localization $\mathrm{Spa}(R, R^+)\left(\frac{f_1,\ldots,f_n}{g}\right)$ to be the subset of $\mathrm{Spa}(R, R^+)$ consisting on those equivalence classes of valuations $x : R \to \Gamma$ such that $|g| \neq 0$ and $|f_i(x)| \leq |g(x)|$ for all $i = 1, \ldots, n$. Then the analytic topology of $\mathrm{Spa}(R, R^+)$ is the topology generated by declaring rational subsets to be open subspaces. We left as an exercise to the reader to prove that for a map $(R, R^+) \to (S, S^+)$ of discrete Huber pairs, the induced map $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$ preserves rational localizations (and so it is continuous for Huber's topology), and that intersections of rational localizations are rational localizations.

The following proposition identifies integrally closed subrings of R with suitable subsets of $\mathrm{Spv}(R)$.

Proposition 5.6.3 ([CS19, Proposition 9.2]). *Let R be a static discrete ring. There is a bijection between integrally closed subrings $R^+ \subset R$ and subsets $U \subset \mathrm{Spv}(R)$ which are intersections of rational localizations $U_{f,1} := \mathrm{Spv}(R)(\frac{f}{1})$. Explicitly, one has*

$$R^+ = \{f \in R : \forall x \in U, |f(x)| \leq 1\}$$

and

$$U = \bigcap_{f \in R^+} U_{f,1}.$$

In particular,

$$R^+ = \{f \in R : \forall x \in \mathrm{Spa}(R, R^+), |f(x)| \leq 1\}.$$

Proof. Let $U \subset \mathrm{Spv}R$ be a subset of the form $U = \varprojlim_f U_{f,1}$, then it is clear that the ring

$$R^+(U) = \{f \in R : \forall x \in U, |f(x)| \leq 1\}$$

Conversely, given an integrally closed subring $R^+ \subset R$ we let $U(R^+) = \bigcap_{r \in R} U_{r,1}$.

A bookkeeping of the definitions shows that for $U \subset \mathrm{Spd}(R)$ as before one has $U = U(R^+(U))$. Conversely, given an integrally closed ring R^+, one easily verifies that $R^+ \subset R^+(U(R^+))$. Now let $f \in R$ be such that for all $x \in U(R^+)$ one has $|f(x)| \leq 1$. If $f \notin R^+$ then f is not in $R^+[\frac{1}{f}] \subset R[\frac{1}{f}]$ and there is a prime ideal $\mathfrak{p}$ of $R^+[\frac{1}{f}]$ that contains $\frac{1}{f}$. Let $\mathfrak{q}$ be a minimal prime contained in $\mathfrak{p}$. We may then find a valuation ring V with a map $\mathrm{Spec}\, V \to \mathrm{Spec}\, R^+[\frac{1}{f}]$ taking the generic point to $\mathfrak{q}$ and the special point to $\mathfrak{p}$. As the image of $\mathrm{Spec}\, R[\frac{1}{f}] \to \mathrm{Spec}\, R^+[\frac{1}{f}]$ contains the minimal prime $\mathfrak{q}$ (the map $R^+[\frac{1}{f}] \to A^+[\frac{1}{f}]$ being injective), we can lift the valuation corresponding to $\operatorname{Spec} V \to \operatorname{Spec} R^+[\frac{1}{f}]$ to $R[\frac{1}{f}]$. The resulting valuation takes values ≤ 1 on R^+, and value > 1 on f since $f \in \mathfrak{p}$. This gives the contradiction. □

Next, we want to show that there is a well defined analytic topology on $\operatorname{Spa}(R, R^+)$ and to construct a structural sheaf.

Proposition 5.6.4. *Let (R, R^+) be a discrete Huber ring and let $(f_1, \ldots, f_n, g)$ be a tuple generating the unit ideal of R. Let $R(\frac{f_1,\ldots,f_n}{g})^+ \subset \pi_0(R[\frac{1}{g}])$ be the integral closure of the subring generated by the image of R^+ and $\frac{f_i}{g}$ for all $i = 1, \ldots, g$. Then the natural map*

$$\Psi : \operatorname{Spa}(R[\frac{1}{g}], R(\frac{f_1, \ldots, f_n}{g})^+) \to \operatorname{Spa}(R, R^+)$$

induces an homemorphism onto $\operatorname{Spa}(R, R^+)\left(\frac{f_1,\ldots,f_n}{g}\right)$. Furthermore, if two rational localizations (R_1, R_1^+) and (R_2, R_2^+) are such that $\operatorname{Spa}(R_1, R_1^+) = \operatorname{Spa}(R_1, R_1^+)$ when considered as subspaces in $\operatorname{Spa}(R, R^+)$, one has a unique isomorphism of R-algebras $R_1 = R_2$ such that $R_1^+ = R_2^+$.

Sketch of the proof. By the description of the adic spectrum in terms of prime ideals and valuation rings of Remark 5.6.2 it is clear that Ψ is injective. To prove that it surjects onto $U = \operatorname{Spa}(R, R^+)\left(\frac{f_1,\ldots,f_n}{g}\right)$, it suffices to see that a point $(\mathfrak{p}, V) \in \operatorname{Spa}(R, R^+)$ belongs to U if and only if g is non zero in $\kappa(\mathfrak{p})$ and $\frac{f_i}{g} \in V$ for all $i = 1, \ldots, n$. But a proof reading of the definitions shows that these are precisely the elements of $\operatorname{Spa}(R[\frac{1}{g}], R(\frac{f_1,\ldots,f_n}{g})^+)$ after identifying prime ideals of both rings via the inclusion $\operatorname{Spec} R[\frac{1}{g}] \subset \operatorname{Spec} R$ using that valuation rings are integrally closed to pass to the integral closure. For proving that it is a homeomorphism for Huber's topology, it suffices to check that a rational localization of $\operatorname{Spa}(R[\frac{1}{g}], R(\frac{f_1,\ldots,f_n}{g})^+)$ is mapped to a rational localization of $\operatorname{Spa}(R, R^+)$, we left this as an exercise to the reader.

Finally, let $(h_1, \ldots, h_k, s)$ be another tuple generating the same rational localization as $(f_1, \ldots, f_n, g)$. First, note that one has an inclusion $F : \operatorname{Spec} R \to \operatorname{Spec}(R, R^+)$ given by mapping a prime ideal $\mathfrak{p}$ to the pair $(\mathfrak{p}, \kappa(\mathfrak{p}))$. It is easy to see that the pre-image along F of a rational localization corresponding to $(f_1, \ldots, f_n, g)$ is $\operatorname{Spec} R[\frac{1}{g}]$. This shows that the localizations $R[\frac{1}{s}]$ and $R[\frac{1}{g}]$ induce the same open Zariski subspaces of $\operatorname{Spec} R$ and so that $R[\frac{1}{s}] = R[\frac{1}{g}]$. Then, the equality

$$R(\frac{f_1, \ldots, f_n}{g})^+ = R(\frac{h_1, \ldots, h_k}{s})^+$$

follows from Proposition 5.6.3, proving the last assertion. □

Definition 5.6.5. Let (R, R^+) be a discrete Huber pair and denote $X = \operatorname{Spa}(R, R^+)$. Let $\mathscr{O}_X$ and $\mathscr{O}_X^+$ be the presheaf on rational open subsets of X mapping $\operatorname{Spa}(R', R'^+)$ to R' and R'^+ respectively.

A final important property of the adic spectrum of discrete Huber rings is that it is immediately sheafy:

Proposition 5.6.6. *Let (R, R^+) be a discrete Huber ring and let $X = \operatorname{Spa}(R, R^+)$ be its adic spectrum. Denote by X_{rat} the site of rational localizations of X. Then the pre-sheaf $\mathscr{O}_X$ on X_{rat} is a sheaf on $\mathscr{D}(R)$ the derived category of R . Similarly, $\mathscr{O}_X^+$ is a sheaf on the abelian category of R^+-algebras*

Proof. There is a continuous map $F : \operatorname{Spec} R \to \operatorname{Spa}(R, R^+)$ mapping a prime ideal $\mathfrak{p}$ to the tuple $(\mathfrak{p}, \kappa(\mathfrak{p}))$. Then the pre-sheaf $\mathscr{O}_X$ is nothing but the (derived) pushforward of the structural sheaf of $\operatorname{Spec} R$, which shows that is a sheaf. The sheaf property for $\mathscr{O}_X^+$ follows since for a rational subspace $U \subset X$ one has

$$\mathscr{O}_X^+(U) = \{f \in \mathscr{O}_X(U) : \forall x \in U, |f(x)| \leq 1\}.$$

□

5.6.2. *Descent for the analytic topology.* In the previous paragraph we have defined the adic spectrum of a discrete Huber pair, define its analytic topology and proved that the structural pre-sheaf is actually a sheaf. Out next goal is to prove that the functor mapping $(R,R^+)\mapsto \mathscr{D}((R,R^+)_{\square})$ satisfies descent for the analytic topology (later we will reinterpret this as !-descent on analytic rings). We need a technical lemma.

Lemma 5.6.7 ([Hub94, Lemma 2.6], [CS19, Lemma 10.4]). *Let (R,R^+) be a discrete Huber pair and $X=\mathrm{Spa}(A,A^+)$. Assume that $U_1,\ldots,U_n\subset X$ are open rational subsets covering X. There there exist $s_1,\ldots,s_N\in A$ generating the unit ideal such that each $X(\frac{s_1,\ldots,s_N}{s_j})$ is contained in some U_i; in particular $\{X(\frac{s_1,\ldots,s_N}{s_j})\}_j$ refines $\{U_i\}$.*

Theorem 5.6.8. *Let (R,R^+) be a discrete Huber ring and let $X=\mathrm{Spa}(R,R^+)$ be its adic spectrum. Let X_{rat} be the site of finite disjoint unions of rational open subspaces of X. Then the functor*

$$\mathscr{D}: X_{\mathrm{rat}}^{\mathrm{op}}\to \mathrm{CAlg}(\mathrm{Pr}^{L,\mathrm{ex}})$$

on presentably symmetric monoidal stable ∞-categories mapping $U\in X_{\mathrm{rat}}$ to $\mathscr{D}((\mathscr{O}_X(U),\mathscr{O}_X(U)^+)_{\square})$ is a sheaf.

Proof. Step 1. Let $\{U_i\}_{i\in I}$ be a finite rational cover of X and let $\{U_J\}_{J\subset I}$ be the poset of finite intersections of the U_i. We want to prove that the natural map

$$\mathscr{D}((R,R^+)_{\square})\to \varprojlim_{U_J\subset X}\mathscr{D}((\mathscr{O}_X(U_J),\mathscr{O}_X(U_J)^+)_{\square}) \tag{5.8}$$

is an equivalence of categories.

By Lemma 5.6.7 we can find elements $s_1,\ldots,s_N$ generating the unit ideal of R such that the rational cover $\{X(\frac{s_1,\ldots,s_N}{s_j})\}_j$ refines $\{U_i\}$. Therefore, it suffices to show descent with respect to the covers of the form $\{U_j=X(\frac{s_1,\ldots,s_N}{s_j})\}_j$.

Step 2. Let us first consider the case when s_N is a unit in R. By replacing s_j by s_j/s_N we can assume that $s_N=1$. We claim that the cover $\{U_j=X(\frac{s_1,\ldots,s_{N-1},1}{s_j})\}_j$ has a refinement given by composition of Laurent covers

$$X=X(\frac{1}{f})\cup X(\frac{f}{1}). \tag{5.9}$$

Indeed, it is refined by the intersection of the Laurent covers $X=X(\frac{1}{s_j})\cup X(\frac{s_j}{1})$ for the different j. Therefore, by an induction argument, it suffices to deal with the case of a single Laurent cover as in (5.9). But this cover corresponds to the maps of analytic rings

$$(R,R^+)_{\square}\to \mathbb{Z}[T]_{\square}\otimes_{\mathbb{Z}[T]}(R,R^+)=(R,R(\frac{f}{1})^+)$$

and

$$(R,R^+)_{\square}\to (\mathbb{Z}[T^{\pm 1}],\mathbb{Z}[T^{-1}])_{\square}\otimes_{\mathbb{Z}[T]}(R,R^+)=(R[\frac{1}{f}],R(\frac{1}{f})^+)$$

where T maps to f. Lemma 5.4.11 implies that these maps give rise to open immersions at the level of locales. Therefore, in order to show descent, by base change along $\mathbb{Z}[T]\to(R,R^+)$ $(T\mapsto f)$, and by Theorem 5.1.6, it suffices to prove that the open localizations $(\mathbb{Z}[T],\mathbb{Z})_{\square}\to\mathbb{Z}[T]_{\square}$ and $(\mathbb{Z}[T],\mathbb{Z})_{\square}\to(\mathbb{Z}[T^{\pm1}],\mathbb{Z}[T^{-1}])_{\square}$ cover the locale $\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\square})$. This amounts to show that the tensor product of the solid $\mathbb{Z}[T]$-idempotent algebras $\mathbb{Z}((T^{-1}))$ and $\mathbb{Z}[[T]]$ vanish. But we have that

$$\mathbb{Z}((T^{-1}))\otimes_{\mathbb{Z}[T]}\mathbb{Z}[[T]]=\mathbb{Z}[[X,T]]^{\mathbb{L}}/(XT-1),$$

but $XT-1$ is a unit in $\mathbb{Z}[[X,T]]$ proving that the tensor vanishes.

Step 3. We now deal with the general case. The elements $s_1, \dots, s_N$ give rise to a Zariski cover $\{\operatorname{Spec} R[\frac{1}{s_j}]\}_j$ of $\operatorname{Spec} R$, we let $V_j = X(\frac{g}{g})$ be the locus where $|g| \neq 0$, and let $\{V_K\}_{K\subset J}$ denote the poset of finite intersections of the V_j's. Note that $R(\frac{g}{g})^+$ is nothing but the integral closure of R^+ in $R[\frac{1}{g}]$. Then, by Corollary 5.5.4 we have that $(R[\frac{1}{g}], R^+)_\square = (R[\frac{1}{g}], R(\frac{g}{g})^+)_\square$. By Zariski descent the natural map

$$\mathscr{D}((R,R^+)_\square) = \varprojlim_{V_K\subset X} \mathscr{D}((\mathscr{O}_X(V_K), \mathscr{O}_X^+(V_K))_\square)$$

is an equivalence. Concretely, the Zariski cover gives rise to a **closed** cover of the locale of $\mathscr{D}(R)$, and by Theorem 5.1.6 the base change to $\mathscr{D}((R,R^+)_\square)$ gives rise to a closed cover as well. By Step 2 the rational cover $\{V_K \cap U_j\}_j$ gives rise an open cover of the locale $\mathscr{L}(\mathscr{D}((\mathscr{O}_X(V_K), \mathscr{O}_X^+(V_K))_\square))$ and so we have

$$\mathscr{D}((\mathscr{O}_X(V_K), \mathscr{O}_X^+(V_K))_\square) = \varprojlim_{U_J\subset X} \mathscr{D}((\mathscr{O}_X(V_K\cap U_J), \mathscr{O}_X^+(V_K\cap U_J))_\square),$$

taking limits with respect to the poset $\{V_K\}$ we get then

$$\begin{aligned}
\mathscr{D}((R,R^+)_\square) &= \varprojlim_{V_K\subset X} \mathscr{D}((\mathscr{O}_X(V_K), \mathscr{O}_X^+(V_K))_\square) \\
&= \varprojlim_{V_K\subset X} \varprojlim_{U_J\subset X} \mathscr{D}((\mathscr{O}_X(V_K\cap U_J), \mathscr{O}_X^+(V_K\cap U_J))_\square) \\
&= \varprojlim_{U_J\subset X} \varprojlim_{V_K\subset X} \mathscr{D}((\mathscr{O}_X(V_K\cap U_J), \mathscr{O}_X^+(V_K\cap U_J))_\square) \\
&= \varprojlim_{U_J\subset X} \mathscr{D}((\mathscr{O}_X(U_J), \mathscr{O}_X^+(U_J))_\square)
\end{aligned}$$

where the first equivalence is Zariski descent for X, the second follows from Step 2 applied to the rational subspaces V_K, the third is a commutation of of limits, and the last is Zariski descent applied to the rational subspaces U_J. This finishes the proof of the theorem. $\square$

In order to define the correct analogue of Corollaries 5.4.3 and 5.4.15 for discrete adic spaces we need to modify Huber's topology on $\operatorname{Spa}(R,R^+)$.

Definition 5.6.9. Let (R,R^+) be a discrete Huber pair. We let $X = \operatorname{Spa}(R,R^+)^{\mathrm{mod}}$ be the adic spectrum endowed with the coarsest topology making the rational localizations of the form $X(\frac{1}{f})$ and $X(\frac{f}{1})$ open ($f\in R$), and the rational localizations of the form $X(\frac{g}{g})$ closed ($g \in R$).

Corollary 5.6.10. *Let (R,R^+) be a discrete Huber pair and let $\operatorname{Spa}(R,R^+)^{\mathrm{mod}}$ be the modified adic spectrum. Then the functor mapping a rational subset $U\subset X$ to $\mathscr{D}((\mathscr{O}_X(U), \mathscr{O}_X^+(U))_\square)$ gives rise to a unique natural surjective map of locales*

$$F : \mathscr{S}(\mathscr{D}((R,R^+)_\square)) \to \operatorname{Spa}(R,R^+)^{\mathrm{mod}}.$$

Proof. This follows from Theorem 5.6.8, the only thing to verify is the surjectivity of F. We want to prove that given $U, U' \subset \operatorname{Spa}(R,R^+)^{\mathrm{mod}}$ two open subspaces for the modified topology, if $F^{-1}(U) = F^{-1}(U')$ then $U = U'$. For this, let $x \in \operatorname{Spa}(R,R^+)$ and let $(\kappa(x), \kappa(x)^+)$ be the residue field at x. We can assume without loss of generality that $U' \subset U$. Then by naturality of F and by taking pullbacks along all affinoid points, it suffices to deal with the case where $R = K$ is a field and $K^+ \subset K$ an integrally closed subring. In this case, the analytic and the modified topology agree, and the open subspaces of $\operatorname{Spa}(K,K^+)$ form a poset by inclusion. Quasi-compact open subspaces of $\operatorname{Spa}(K,K^+)$ are of the form $\operatorname{Spa}(K,K^{',+})$ for $K^+\subset K^{',+}$ an integrally closed ring. Thus, by taking intersections of U with open affinoids, to prove surjectivity it suffices to show that for $U \subset \operatorname{Spa}(K,K^+)$ an arbitrary open subspace, if $F^{-1}(U) = F^{-1}(\operatorname{Spa}(K,K^+))$ then $U = \operatorname{Spa}(K,K^+)$. Suppose this does not hold and let $x\in \operatorname{Spa}(K,K^+)$ be the maximal closed point, then $F^{-1}(x)=\emptyset$. Let A_x be the idempotent algebra corresponding to x, and let $\{Z_j\}_j$ be the poset of closed subspaces with qcqs open complements with idempotent algebras A_{Z_j}. Then we have that

$$0 = A_x = \varinjlim_j A_{Z_j}.$$

Since the colimit is filtered, there exists some j such that $1\in A_{Z_j}$ is zero and so $A_{Z_k}=0$ for all $k\geq j$. But this will imply that $F^{-1}(X\backslash Z_j)=F^{-1}(X)$ and $X\backslash Z_j=\mathrm{Spa}(K,K^{',+})$ is affinoid with K^+ strictly contained in $K^{',+}$. This contradicts the fully faithfulness of discrete Huber pairs into analytic rings of Proposition 5.5.5, proving what we wanted. □

Remark 5.6.11. We have not stated Corollary 5.6.10 for general discrete adic spaces since the analytic topology differs from the *modified topology*, so that discrete adic spaces are not obtained from gluing affinoid spaces along the modified topology. Instead, since the analytic topology contains both closed and open subspaces of the modified topology, discrete adic spaces will be examples of gluing affinoids via !-covers (to be defined later in the notes).

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