# STUDY GROUP ON DERIVED DEFORMATION THEORY

## 1. Overview

Deformation theory is a standard and fundamental tool in algebraic and arithmetic geometry that has been used in a broad type of contexts for studying deformations of algebro-geometric objects such as proper and flat maps of schemes or Galois representations. As it is stated in [Lur11], the main thesis in deformation theory in characteristic zero is that "the formal completion of a moduli problem $X$ at a point $x$ is governed by a differential graded Lie algebra that ought to be the (stabilization of) the tangent space of $X$ at $x$". The ideas of relating formal moduli problems with differential graded algebras dates back to a letter of Drinfeld [Dri14], and was formalized in a concrete theorem independently by Pridham [Pri11] and Lurie [Lur11].

The objective of this study group is to give an introduction to deformation theory in derived algebraic geometry from the point of view of [Lur11]. More precisely, the main goal is to introduce the category of formal moduli problems Moduli for connective dg algebras over a field $k$ in characteristic zero (eq. connective $\mathbb{E}_\infty$ $K$-algebras, eq. animated $K$-algebras); the category of graded differential Lie algebras $\mathrm{Lie}_K^{\mathrm{dg}}$ as well as its associated $\infty$-category $\mathrm{Lie}_K$; and finally to prove the main theorem consisting of the equivalence of $\infty$-categories

$$\Psi : \mathrm{Lie}_K^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Moduli}.$$

Throughout the course of the study group we also expect to see different concrete examples that allow a better understanding of the functor $\Psi$.

In order to take a safe approximation for the distribution of talks we shall follow part of the lectures on $\infty$-categories and deformation theory of Lukas Brantner [Bra24], and then move to [Lur11] for the main theorem. We also highlight the following additional references:

- [Qui69] for a first instance of the equivalence between subcategories of dg algebras and dg Lie algebras arising from topology.
- [Man22] for a classical approach to deformation theory and its relation with dg Lie algebras.
- [BM23] for a positive and mixed characteristic versions of Lurie-Pridham theorem.

## 2. Talks

2.1. **Overview talk.** Introductory talk.

2.2. **Background in $\infty$-categories.** The speaker will give an informal introduction to $\infty$-categories. The references are Section 1.1 of these lecture notes by Adeel Khan, Section 1.1.1 and 1.1.2 of [Lur09], Lecture 2 of [Bra24], and Lecture 2 of [Bra22]. The speaker should give a rough idea of what an infinity category is, and focus more in the operational way to deal with them. For example, he or she should introduce the category of spaces or anima, functor categories, limits and colimits. As example, the speaker should mention the derived $\infty$-category of $K$-vector spaces for $K$ a field.

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2.3. **Animated and commutative rings.** The speaker should introduce the notion of animation of [Bra24, Lecture 5] or Section 1.2 of Adeel Khan's notes (see also [CS23, Section 5.1.4] and [CS20, Section 11.1]), in particular the definition of animated algebras and animated abelian groups. On the other hand, the speaker should introduce (symmetric) monoidal $\infty$-categories, the notion of (commutative algebra) in a (symmetric) monoidal category, and modules over the algebra. The speaker could use as working example to the case of commutative algebras in the $\infty$-derived category of $K$-vector spaces for a field $K$ in characteristic zero; see [Bra24, Lecture 2], [Bra22, Lecture 3], [Lur17, Section 1.3.1] and [Lur04, Section 2]. A good example to include is [BS22, Remark 2.5].
**Speaker**: Sangmin Ko

2.4. **Barr-Beck-Lurie theorem.** In this talk the speaker will explain the statement of the Barr-Beck-Lurie theorem as in Theorem 3.11 of [Bra24, Lecture 3]. For that, the speaker will introduce monads, give an informal definition of an adjunction and explain how they produce monads (see also [Bra24, Lecture 1 Section 1.3]). The speaker should also say what a totalization an geometric realization is, and finally state the Barr-Beck-Lurie theorem. As examples she or he could explain the adjunction between animated rings and anima, commutative an associative algebras on dg $K$-algebras. The references for this talk are [Bra24, Lecture 3], [Bra22, Lecture 4] and [Lur04, Section 2].

2.5. **Moduli problems and the tangent complex.** The goal of this talk is to introduce the category of small $K$-algebras, the category of formal moduli problems, and the tangent complex as $K$-linear spectrum. The references for this talk are [Lur11, Sections 1.1 and 1.2]. The material in the references are presented in an abstract set up, the speaker is encouraged to restrict to the case of small $K$-algebras. In particular, the speaker should discuss the different equivalent definitions of formal moduli problem of Propositions 1.1.15 and 1.1.19 of [Lur11, Section 1.1]. He or she should define the tangent complex of a formal moduli problem and show that it has a structure of spectrum (see Proposition 1.2.4), and that the functor from formal moduli problems to spectrum given by the tangent space is conservative (Proposition 1.2.10). The speaker can finish with a crude version of the main theorem by saying that the functor mapping a moduli problem to the tangent complex is monadic, see last paragraph of[Lur11, Section 1.3].
**Speaker**: Nicolás Vilches Reyes

2.6. **Differential graded Lie algebras.** The speaker will introduce the notion of differential graded Lie algebras (dgla) $\mathrm{Lie}_K^{dg}$ over a field $K$ of characteristic 0 following [Lur11, Section 2.1]. In particular, the speaker should explain the enveloping algebra construction, mention the model structure on dgla algebras used to define the $\infty$-category $\mathrm{Lie}_K$, explain the adjunction between dgla and associative rings, and explain that the forgetful functor from dgla to $K$-linear spectrum is monadic.

2.7. **Construction of the Koszul duality functor.** The speaker will discuss the cohomology of dgla (also called the Eilenberg-Mac Lane spectrum) and show that it gives rise to a commutative $K$-algebra (or commutative differential graded algebra) following [Lur11, Section 2.2]. In particular, she or he should present Proposition 2.2.12, Construction 2.2.13 and Proposition 2.2.17. Then, following the beginning of [Lur11, Section 2.3] the speaker should deduce the Koszul duality functor $\mathcal{D}$ and state the precise equivalence between Moduli and dgla of Theorem 0.0.13 (see the discussion after the theorem for the precise functor).

2.8. **Proof of the main Theorem I: properties of the Koszul duality functor.** This talk concerns the first part of the proof of the main theorem (Theorem 1.3.12 of [Lur11]). The goal of the speaker is to show Theorem 2.3.1 of [Lur11], see Definition 1.3.9 the notion of "deformation theory". For this, the key tool is Proposition 2.3.4 and Lemma 2.3.5. The speaker should also prove Corollary 2.3.6 identifying which dgla correspond to pro-representable formal moduli problems (Definition 1.5.3), and that that the Koszul duality functor has as underlying object the tangent complex as in Proposition 2.3.9.

**Speaker**: Ivan Zelich

2.9. **Proof of the main Theorem II: smooth hypercovers.** This is the second part of the proof of the main theorem (Theorem 1.3.12 of [Lur11]), and the one that involves more geometry than algebra. The reference for this talk is [Lur11, Section 1.3-1.5]. The speaker should recall the pro-representable objects (Definition 1.5.3) and define the notion of formally smoothness (Proposition 1.5.5). Then, he or she should explain the content of Proposition 1.5.8 assuming its proof, and finally show Theorem 1.3.12.

2.10. **Quasi-coherent sheaves.** This is a bonus talk that could take place depending on time. The speaker should explain the relation between quasi-coherent sheaves of the formal moduli problems and the $\infty$-category of modules over the associated dgla. The reference for this talk is [Lur11, Section 2.4].

## References


[BM23] Lukas Brantner and Akhil Mathew. Deformation theory and partition Lie algebras, 2023.

[Bra22] Lukas B. Brantner. $\infty$-categories in algebraic geometry, 2022.

[Bra24] Lukas B. Brantner. $\infty$-categories and deformation theory, 2024.

[BS22] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[CS20] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry. `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`, 2020.

[CS23] Kestutis Cesnavicius and Peter Scholze. Purity for flat cohomology, 2023.

[Dri14] V. Drinfeld. A letter from Kharkov to Moscow. *EMS Surv. Math. Sci.*, 1(2):241–248, 2014. Translated from the Russian by Keith Conrad.

[Lur04] Jacob Lurie. Derived Algebraic Geometry, 2004.

[Lur09] Jacob Lurie. *Higher Topos Theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur11] Jacob Lurie. Derived Algebraic Geometry X: Formal moduli problems, 2011.

[Lur17] Jacob Lurie. Higher algebra. 2017.

[Man22] Marco Manetti. *Lie methods in deformation theory*. Springer Monographs in Mathematics. Springer, Singapore, [2022] ©2022.

[Pri11] J. P. Pridham. Corrigendum to "Unifying derived deformation theories" [Adv. Math. 224 (3) (2010) 772–826] [mr2628795]. *Adv. Math.*, 228(4):2554–2556, 2011.

[Qui69] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.