NOTES ON SOLID GEOMETRY

JUAN ESTEBAN RODRÍGUEZ CAMARGO

Abstract. These are notes of a seminar held in Columbia university during the Spring of 2024 about the new theory of analytic stacks of Clausen and Scholze. The seminar is inspired from the Lecture Series of Analytic Stacks, all results are due to Clausen and Scholze, and any mistake or misconception is completely due to the author.

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1. Introduction

Different geometric theories appear all across mathematics: differentiable manifolds, complex and real analytic varieties, rigid analytic spaces, adic spaces, Berkovich spaces, algebraic varieties and schemes, formal schemes, etc. The aim of "analytic stacks" is to define a general ecosystem where the previous (and many more!) "theories of analytic and algebraic geometries" cohabit and interact each other. To motivate the distribution of future talks let us make explicit the obstructions that mathematicians have face all over the years when dealing with analytic geometry, and how condensed mathematics and analytic stacks have solved these issues.

1.1. Light condensed sets. The building blocks in theories such as algebraic varieties or schemes consist simply of commutative rings satisfying some additional algebraic properties. This leads to a pleasant treatment of geometry that is studied in purely algebraic terms. However, in other theories such as differentiable manifolds and complex or rigid analytic varieties, the building blocks turn out to be some sort of topological rings, more often Banach or Fréchet rings. Then, any general form of "analytic geometry" that inherits a similar formalism as algebraic geometry must be built up over an algebraic theory of "topological rings". However, history has shown that the datum of a topology does not mixes very well with that of an algebraic structure. A very simple and clever solution to this is provided by condensed mathematics [CS19], where "topology" is changed by the topos of (light) condensed sets. Therefore, our first replacement for topological "preferred algebraic structure" (eg. ring/module/abelian group/monoid) will be condensed "preferred algebraic structure".

The idea behind condensed mathematics follows the philosophy of Grothendieck saying that a space $X$ must be studied by looking at maps $Y \rightarrow X$ from some "test objects" $Y$. For this approach to be useful, one needs to choose the "test objects" wisely. In our situation, we want to study (reasonable) topological spaces, and a first class of reasonable topological spaces are compact Hausdorff spaces. It turns out that compact Hausdorff spaces can be reconstructed from a certain class of "very acyclic" spaces. Concretely, let Prof be the category of profinite sets/totally disconnected
compact Hausdorff spaces. We endow $\text{Prof}$ with the Grothendieck topology whose covers are given by finitely many jointly surjective maps. As justification for this choice, recall that any surjective map of compact Hausdorff spaces is a quotient map, and that any compact Hausdorff space $X$ admits a surjection from a profinite set. For instance, the closed interval $[0,1]$ admits a surjective map from the Cantor set $\prod_\mathbb{N}\{0,1\} \to [0,1]$ by sending a sequence $(a_n)$ to the real number written in binary decimals $(a_n) \mapsto 0.a_1a_2a_2\cdots$.

**Definition 1.1.1.** A condensed set is a sheaf $T : \text{Prof}^{\text{op}} \to \text{Set}$ (modulo some set-theoretical technicalities i.e. accessible), we let $\text{CondSet}$ denote the category of condensed sets. For $X$ a (reasonable) topological space (eg. Hausdorff), we define its condensification $X \in \text{CondSet}$ by taking

$$X(S) = \text{Map}(S, X)$$

the space of continuous maps from $S$ to $X$, with $S \in \text{Prof}$.

Most of the spaces we care of in topology (such as countably generated CW complexes), geometry (eg. manifolds), and analysis (eg. Banach, Fréchet spaces) are endowed with a topology for which understanding converging sequences is often enough. More precisely, the most interesting topological spaces are (locally) metrizable. Thus, a good balance in condensed mathematics between capturing all the relevant information and avoiding unnecessary technicalities is given by light condensed sets:

**Definition 1.1.2.** A light profinite set is a metrizable profinite set, we $\text{Prof}^{\text{light}}$ be the category of light profinite sets. A light condensed set is a sheaf $T : \text{Prof}^{\text{light,op}} \to \text{Set}$, we let $\text{CondSet}^{\text{light}}$ denote the category of light condensed sets.

Finally, for any algebraic structure $\mathcal{C}$ (aka. a category with small limits and colimits), its condensation $\text{Cond}(\mathcal{C})$ is the category of sheaves $T : \text{Prof}^{\text{light,op}} \to \mathcal{C}$ from light profinite sets in $\mathcal{C}$. For example, we can talk about (light) condensed abelian groups, rings, monoids, etc. The notion of light condensed "preferred algebraic structure" is the replacement we shall use for its topological analogue.

### 1.2. Analytic rings

As it was mentioned before, part of the datum of the building blocks in a general theory of analytic geometry must involve some kind of topological (aka condensed) ring. On the other hand, the most fundamental invariant of a space $X$ in both analytic or algebraic geometry is its category of (quasi-)coherent sheaves $\text{QCoh}(X)$. In classical algebraic geometry this category is obtained by gluing, using "Zariski descent", the category of modules $\text{Mod}(A)$ of commutative rings $A$. However, in the case of complex and rigid geometries, the best that one can (classically) do in a systematic and algebraic manner is to built up the category of coherent modules $\text{Coh}(X)$, imposing in this way finiteness conditions to the sheaves living over $X$. In particular, for a general morphism $f : X \to Y$ of rigid or complex analytic spaces, the sheaf $f_*\mathcal{O}_X$ does not belong to the category attached to $X$. On the other hand, even though condensed rings are some kind of topological rings, in analytic geometries we often want to have some kind of "complete tensor product" and a category of "complete modules". It turns out that if $A$ and $B$ are two condensed rings, then the underlying ring of $A \otimes_Z B$ is just the algebraic tensor $A(*) \otimes_Z B(*)$, proving that we still need to do something else.

The notion of analytic ring appears as a solution to the previous problematics. The datum of an analytic ring $A$ consists of a condensed ring $A^\circ$ and a stable $\infty$-category $\mathcal{D}(A)$ of "complete $A$-modules". Before enumerating the features of $\mathcal{D}(A)$, let us do a brief detour explaining this jump from an abelian category of modules to a stable $\infty$-category: in classical algebraic geometry, the category $\text{QCoh}(X)$ of quasi-coherent sheaves is endowed with a symmetric tensor product $\otimes_{\mathcal{O}_X}$. Within this tensor product one can construct fiber products $X \times_Y Z$ of (affine) schemes by simply taking the (affine) scheme represented by the tensor product of rings. However, when dealing with
cohomological invariants of algebraic varieties, it is natural to enter the world of derived categories. In this realm the "correct fiber product" $X \times_Y Z$ is not longer constructed using the "abelian" tensor product of rings but instead the "derived tensor product". Thanks to the current status of higher category theory and higher algebra, eg. \[lur09\] \[lur17\] \[lur18\], we have nowadays the categorical tools to develop theories of "derived algebraic geometries" as in \[lur04\] \[lur18\].

In the former theory of analytic geometry, classical abelian or triangulated categories of quasi-coherent sheaves are not enough to obtain descent and glue to more general spaces (a reason is the lack of "complete" flatness even for some simple maps such as open immersions of rigid or complex analytic spaces). Instead, stable $\infty$-categories are perfectly suited for these purposes. As consequence of the previous explanation, the general theory of analytic rings depends in higher categorical foundations (eg. the underlying condensed ring $A^\infty$ should be an animated or a condensed $\mathbb{E}_\infty$-ring), even though the most fundamental examples still can be explained in the world of abelian categories. For the reader that is not comfortable with the language of higher category theory, we recommend to consider $\mathcal{D}(A)$ as a classical derived category in a first approach, and accept some features of $\infty$-derived categories for granted such as the existence of arbitrary (small) limits of $\infty$-categories \[lur09\] \[§3.3.3\], or the adjoint functor theorem \[lur09\] Corollary 5.5.2.9.

Going back to the category $\mathcal{D}(A)$, it ought satisfy the following properties:

1. It should be a full subcategory $\mathcal{D}(A) \subset \mathcal{D}(A^\infty)$ of the derived $\infty$-category of condensed $A^\infty$-modules stable under all limits and colimits, and "tensored over condensed abelian groups". This are the basic requirements for doing homological algebra over $A$.

2. There is a "completion functor" $A \otimes_{A^\infty} - : \mathcal{D}(A^\infty) \to \mathcal{D}(A)$, left adjoint to the natural inclusion (note that we have dropped derived decorations in the tensor). Moreover, $\mathcal{D}(A)$ can be uniquely promoted to a symmetric monoidal category such that $A \otimes_{A^\infty} -$ is a symmetric monoidal functor. Similarly as for schemes, we require our category of modules to be endowed with a "complete tensor product" that will generalize "complete tensor products" in classical theories of analytic geometries.

3. The completion functor $A \otimes_{A^\infty} -$ should preserve connective objects: $A \otimes_{A^\infty} - : \mathcal{D}(A^\infty)_{\geq 0} \to \mathcal{D}(A^\infty)_{\geq 0}$. This will endow $\mathcal{D}(A)$ with a $t$-structure arising from condensed $A^\infty$-modules.

4. We have $A^\infty \in \mathcal{D}(A)$ (we want our topological ring to be complete!).

**Definition 1.2.1.** An analytic ring $A$ is a pair $(A^\infty, \mathcal{D}(A))$ consisting on a light condensed animated ring $A^\infty$, and a full subcategory $\mathcal{D}(A) \subset \mathcal{D}(A^\infty)$ of "complete modules" satisfying properties (1)-(4) above. A morphism of analytic rings $f : A \to B$ is a morphism of condensed rings $A^\infty \to B^\infty$ such that the forgetful functor $f_* : \mathcal{D}(B^\infty) \to \mathcal{D}(A^\infty)$ sends $\mathcal{D}(B)$ to $\mathcal{D}(A)$. We let $\text{AnRing}$ denote the $\infty$-category of analytic rings.

It turns out that $\text{AnRing}$ a is a presentable $\infty$-category (cf. \[lur09\] \[§5.5\] for the notion of presentability), in particular it admits all (small) colimits (cf. \[CS20\] Proposition 12.12 and \[Man22\] Proposition 2.3.15]). Analytic rings shall be the bulding blocks in the theory of analytic stacks.

### 1.3. **Analytic stacks.** Let $\text{Ring}$ be the category of rings. Schemes are constructed out from $\text{Ring}$ by gluing using the Zariski topology. In particular, a scheme can be seen as an object in $\text{Sh}_{\text{Zar}}(\text{Ring}^{\text{op}}, \text{Set})$, i.e. a sheaf for the Zariski topology in the opposite category of rings, aka, affine schemes. Similarly algebraic spaces (resp. Artin stacks) are obtained by "gluing affine schemes" along étale or smooth maps, they then define sheaves in more refined Grothendieck topologies such as the étale or flat topologies. Moreover, when defining stacks in derived algebraic geometry \[lur04\], it is mandatory to not just consider functors with values in sets but in anima $\text{Ani}$ (aka. $\infty$-groupoids or spaces).

For the theory of analytic stacks we want to define a suitable Grothendieck topology $G$ on $\text{AnRing}$ such that "analytic stacks" are given by (hyper)sheaves

$$\text{AnStack} = \text{Sh}_G(\text{AnRing}^{\text{op}}, \text{Ani}).$$
The question that arises is which Grothendieck topology should we consider? Well, by definition analytic rings are not just its underlying condensed ring but its category of modules. Indeed, an analytic ring is (essentially) completely determined by its category of modules! Thus, whatever Grothendieck topology we choose, the functor $A \mapsto \mathcal{D}(A)$ should certainly satisfy descent. On the other hand, we want a refined enough Grothendieck that explains already existing "identifications" from classical analytic geometries:

Let $\mathbb{Q}_p$ be the field of $p$-adic numbers, and consider the projective space $\mathbb{P}^1_{\mathbb{Q}_p}$. There are different ways to construct $\mathbb{P}^1_{\mathbb{Q}_p}$. First, we have the algebraic geometry manner that glues the (spectrum of the) rings $\mathbb{Q}_p[T]$ and $\mathbb{Q}_p[T^{-1}]$ along the intersection $\mathbb{Q}_p[T^{\pm 1}]$. On the other hand, we have rigid geometry and we can construct $\mathbb{P}^1_{\mathbb{Q}_p}$ by gluing the (adic spectrum of the) Tate algebras $\mathbb{Q}_p\langle T \rangle$ and $\mathbb{Q}_p\langle T^{-1} \rangle$ along the intersection $\mathbb{Q}_p\langle T^{\pm 1} \rangle$. Thus, we want the theory of analytic stacks to be able to identify these both constructions of $\mathbb{P}^1_{\mathbb{Q}_p}$ as the same space, getting as a result a geometric version of GAGA theorems.

In later talks we shall introduce the formal definition of the Grothendieck topology used for defining analytic stacks. A key tool in its definition will be the abstract theory of six functor formalisms built for analytic rings.

1.4. Examples. During the introduction of light condensed sets, analytic rings and analytic stacks, we shall study in more detail some examples arising from algebraic geometry and the theory of adic spaces (solid theory). We will just shortly mention the existence and some features of archimedean and global examples of analytic rings (liquid and gaseous theory).

Solid abelian groups. Let $\text{CondAb}^\text{light}$ denote the category of light condensed abelian groups. We shall define the subcategory of (light) solid abelian groups $\text{Solid} \subset \text{CondAb}^\text{light}$ by imposing a condition extracted from the idea that "converging sequences in non-archimedean analysis are precisely the null sequences". The category of solid abelian groups is endowed with a tensor product that we denote by $\otimes$, it has $\mathbb{Z}$ as unit, and so it defines and analytic ring $\mathbb{Z}$ that we call the "solid integers". The category Solid has a compact projective generator $\prod_N \mathbb{Z}$ that is flat for $\otimes$, and satisfies

$$
\prod_I \mathbb{Z} \otimes_{\mathbb{Z}} \prod_J \mathbb{Z} = \prod_{I \times J} \mathbb{Z}
$$

for countable sets $I, J$. This category is completely disjoint from archimedean analysis, namely, the solidification of $\mathbb{R}$ is just $0$. Examples of solid abelian groups are discrete groups, $p$-adically complete modules, $\mathbb{Q}_p$-Banach and Fréchet spaces, etc. It also holds that (most) of the completed tensor products appearing in non-arhipimedean geometry coincide with $\otimes$ (eg. $p$-complete tensor products of Banach spaces, projective tensor product of nuclear Fréchet spaces).

Liquid vector spaces. Let $q \in (0, 1]$. The analytic ring of liquid real vector spaces was constructed in \cite{CS20}. The construction of this analytic ring requires a lot of effort due to the non-locally convex functional analysis involved. For instance, if $\mathbb{R}_{<q}$ denotes the analytic ring of $<q$-liquid real numbers, and $S$ is a profinite set, then the free liquid real vector space $\mathbb{R}_{<q}[S]$ is not the naive guess of signed Radon measures on $S$, but a certain space of $(<q)$-convex Radon measures. The liquid tensor product agrees with the projective tensor product for nuclear Fréchet spaces, as well as for their duals, see \cite{CS22} IV.

Gaseous rings. As we shall see later, one of the main advantages of the new foundations for the theory of analytic rings, based on light condensed sets, is that it is much easier to construct analytic rings out from inverting some concrete maps of modules. The difficulty is then translated in computing the functors of "measures" $A[S]$ for $S \in \text{Prof}^\text{light}$. The gaseous ring stack is defined in this way via some universal property in the category of analytic rings. It specializes in both solid and liquid stacks, and its underlying ring $\mathbb{Z}[\varphi]^\text{gas} \subset \mathbb{Z}[\varphi]$ consists on power series of at most
polynomial growth:

\[ Z[q]^{gas}(\ast) = \{ \sum_{n \gg -\infty} a_n q^n : \exists m, k > 0 \text{ such that } \lim_{n \to \infty} |a_n|(n+m)^{-k} = 0 \}. \]

The gaseous ring was motivated from the construction of Tate’s elliptic curve \( \mathbb{G}_{m,A}^\text{an}/q^\mathbb{Z} \) in an universal way.

2. Light condensed mathematics

In this talk we will study the basics in light condensed mathematics, this involves light profinite sets, light condensed sets and light condensed abelian groups.

2.1. Light profinite sets. Condensed mathematics proposes a better algebraic framework that replaces topological spaces, namely condensed sets. The building blocks of condensed sets are profinite sets that we briefly recall down below.

**Proposition 2.1.1.** The following categories are equivalent.

1. The pro-category of finite sets \( \text{Pro}(\text{Fin}) \) where maps are given by
   \[
   \text{Map}(\lim_i S_i, \lim_j T_j) = \lim_j \lim_i \text{Map}(S_i, T_i).
   \]

2. The category of totally disconnected compact Hausdorff spaces with continuous maps.

3. The opposite category of Boolean algebras.

We let \( \text{Prof} \) denote the category of profinite sets, considered as in (1) or (2) above.

**Proof.** We just construct the equivalences. From (1) to (2) we take a projective system \( \{S_i\}_i \) and pass to the topological space \( S = \lim_i S_i \) endowed with the limit topology. From (2) to (1) we take a totally disconnected compact Hausdorff space and consider the projective system \( \{S_i\}_{i \in I} \) of finite quotients of \( S \), equivalently, the projective system of partitions of \( S \) in clopen subspaces. From (2) to (3) we take a totally disconnected compact Hausdorff space \( S \) and consider the Boolean algebra \( A = C(S, \mathbb{F}_2) \) of continuous functions from \( S \) to \( \mathbb{F}_2 \). From (3) to (2) we take a Boolean algebra \( A \) and consider its spectrum \( \text{Spec} A \) as a topological space. \( \square \)

A delicate issue when working with the category of all profinite sets is that it is not essentially small, i.e. there is not a set of isomorphism classes of objects. On the other hand, all the spaces we actually care about appearing in geometry, topology or analysis (such as manifolds, CW complexes, Banach or Fréchet spaces) admit a norm, and can be recovered within a set of smaller profinite sets.

**Proposition 2.1.2.** Let \( S \) be a profinite set, the following are equivalent.

1. \( C(S, \mathbb{Z}) \) is countable
2. \( S \) is metrizable
3. \( S \) is 2-countable
4. \( S \) can be written as a sequential limit of finite sets.

**Proof.** Urysohn’s metrization theorem implies that a compact Hausdorff space is metrizable if and only if it is 2-countable, this shows (2) \(\iff\) (3).

(3) \(\iff\) (4). By Proposition 2.1.1 the passage from a totally disconnected compact Hausdorff space \( S \) to a projective system of finite sets is made by taking the system of partitions of \( S \) into clopen subspaces, since \( S \) is 2-countable this projective system is countable. Conversely, if \( S = \lim_{n \to \infty} S_n \) is a sequencial limit of finite sets, taking the fibers of the maps \( S \to S_n \) defines a countable basis for the topology of \( S \).

(4) \(\Rightarrow\) (1). If \( S = \lim_{n \to \infty} S_n \), then \( C(S, \mathbb{Z}) = \lim_{n \to \infty} C(S_n, \mathbb{Z}) \) which is countable.
\begin{proof}
We let \( S \rightarrow \) denote the category of light profinite sets.

Proposition 2.1.5. Let \( S \) be a light profinite set and let \( U \subset S \) be an open subspace. Then \( U \) is a countable disjoint union of light profinite sets.

\begin{proof}
Let us write \( S = \lim_{\leftarrow n} S_n \) and let \( Z = \lim_{\leftarrow n} Z_n \) with \( Z_n \subset S_n \) the image of \( Z \) in \( S_n \). Let \( \pi_n : S \rightarrow S_n \) and \( \pi_{n,m} : S_m \rightarrow S_n \) denote the projection maps. We define \( Y_0 = S_0 \setminus Z_0 \) and for \( n \geq 1 \) we let \( Y_n = S_n \setminus (Z_n \cup \pi_{n-1, n-1}^{-1}(Y_{n-1}) \cup \cdots \cup \pi_{n,0}^{-1}(Y_0)) \). Then \( U = \bigsqcup_{n \in \mathbb{N}} \pi_n^{-1}(Y_n) \).
\end{proof}

Proposition 2.1.6. Let \( S \) be a light profinite set. Then \( S \) is an injective object in \( \text{Prof}^{\text{light}} \).

\begin{proof}
Let \( f : X \rightarrow Y \) be an injection of light profinite sets and let \( g : X \rightarrow S \) be a map. The map \( f \) is a closed immersion, then we can write it as a sequential limit \( \lim_{\leftarrow n} (f_n : X_n \rightarrow Y_n) \) of injective finite sets. We can write the map \( g \) as a sequential limit of finite sets \( \lim_{\leftarrow n} (g_n : X_k_n \rightarrow S_n) \) with \( k_n \) some increasing sequence. After taking a subsequence we can assume that \( k_n = n \). Then, we can always find a map \( h_0 : Y_0 \rightarrow S_0 \) extending \( g_0 \), and provided the extension \( h_n : Y_n \rightarrow S_n \), we can always find a map \( h_{n+1} : Y_{n+1} \rightarrow S_{n+1} \) extending \( g_{n+1} \) that reduces to \( h_n \) in the \( n \)-th step. Taking the limit \( h = \lim_{\leftarrow n} h_n \) we get the desired map \( h : Y \rightarrow S \) extending \( g \).
\end{proof}

Proposition 2.1.7 (\cite[Theorem 5.4]{[CS19]}). Let \( S \) be a light profinite set, then the space of continuous functions \( C(S, \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module.

\begin{proof}
Let us write \( S = \lim_{\leftarrow n} S_n \) as a sequential limit with surjective maps. We can find compatible sections 
\( S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \)
and then inductively find compatible sections \( S_0 \rightarrow S, S_1 \rightarrow S, \ldots \). Then, we know that
\[
C(S, \mathbb{Z}) = \lim_{\rightarrow n} C(S_n, \mathbb{Z}),
\]
and we just found compactible sections of \( C(S_n, \mathbb{Z}) \rightarrow C(S, \mathbb{Z}) \), since the modules \( C(S_n, \mathbb{Z}) \) are free, this shows that \( C(S, \mathbb{Z}) \) is also free.
\end{proof}

Example 2.1.8. The two examples of light profinite sets that will be the most relevant for us:

(1) The one point compactification of \( \mathbb{N} \), namely, \( \mathbb{N} \cup \{\infty\} \). It can be written as 
\[
\mathbb{N} \cup \{\infty\} = \lim_{\leftarrow n} \{1, 2, \ldots, n, \infty\}
\]
where for \( m \geq n \) the map \( \{1, 2, \ldots, m, \infty\} \rightarrow \{1, 2, \ldots, n, \infty\} \) sends all the elements \( k \geq n + 1 \) to \( \infty \).
(2) The Cantor set \( S = \prod_{n \in \mathbb{N}} \{0, 1\} \), it admits a surjective map onto the interval \([0, 1]\) by taking binary decimal expansions.

The relevance of the Cantor set is explained in the following proposition.

**Proposition 2.1.9.** A profinite set is light if and only if it admits a surjective map from the Cantor set.

**Proof.** Let \( S = \lim_{\leftarrow n} S_n \) be a light profinite set, and let us suppose that \( S \to S_n \) is surjective for all \( n \). Then, we can always find a sequence of non-negative integers \((k_n)_{n \in \mathbb{N}}\) and compatible surjection maps for varying \( n \)

\[
\prod_{m=0}^{k_n} \{0, 1\} \to S_n.
\]

Taking the limit we get the desired surjection from the Cantor set. \( \square \)

2.2. **Light condensed sets.** After the previous preparations of light profinite sets we can finally define light condensed sets (cf. [CS19, Definition 1.2]):

**Definition 2.2.1.** A light condensed set is a sheaf in the category of light profinite sets for the Grothendieck topology given by finite disjoint unions of jointly surjective maps. More concretely, a condensed set is a functor \( T : \text{Prof}^{\text{light,op}} \to \text{Set} \) such that

1. \( T(\emptyset) = \ast \).
2. \( T(S_1 \sqcup S_2) = T(S_1) \times T(S_2) \).
3. For all surjective map \( S_1 \to S_2 \) we have

\[
T(S_2) = \text{eq}(T(S_2) \Rightarrow T(S_2 \times S_1 \times S_2)).
\]

We let \( \text{CondSet}^{\text{light}} \) denote the category of light condensed sets.

**Remark 2.2.2.** By Proposition 2.1.4, sequential limits of covers in \( \text{Prof}^{\text{light}} \) are covers. In particular, the topos of condensed sets is replete in the sense of [BS14, §3], namely, sequential limits \( T = \lim_{\leftarrow n} T_n \) of condensed sets with surjective maps are still surjective. Indeed, by definition of the Grothendieck topology, given \( S_0 \to T_0 \) an \( S_0 \)-point of \( T_0 \) there is a surjective map \( S_1 \to S_0 \) and a lift \( S_1 \to T_1 \). Repeating this process we find a compatible sequence of points \( S_n \to T_n \) with \( S_{n+1} \to S_n \) a surjective map. Then, taking limits \( S = \lim_{\leftarrow n} S_n \to T \), we get a lift of \( S_0 \to T_0 \) to \( S \to T \) and the map \( S \to S_0 \) is a cover in the Grothendieck topology being surjective by Proposition 2.1.4.

**Example 2.2.3.**

1. Let \( T \) be a light condensed set, then the set \( T(\mathbb{N} \sqcup \{\infty\}) \) is heuristically the space of convergence sequences with fixed limit, namely, this is exactly the case when \( T \) arises from the condensification of a topological space. If \( T = X \) arises from a Hausdorff space then the set of convergence sequences are determined by its restriction to \( \mathbb{N} \), i.e. the map \( T(\mathbb{N} \sqcup \{\infty\}) \to T(\mathbb{N}) \) is injective. In general, a convergence sequence can have different limits, so the map \( T(\mathbb{N} \sqcup \{\infty\}) \to T(\mathbb{N}) \) is not necessarily injective.

2. Let \( \text{Top} \) denote the category of topological space. We define the condensification functor

\[
(\_)_c : \text{Top} \to \text{CondSet}^{\text{light}}
\]

mapping a topological space \( X \) to the condensed set \( \_X : S \mapsto C(S, X) \) for \( S \in \text{Prof}^{\text{light}} \).

3. The Yoneda embedding \( \text{Prof}^{\text{light}} \to \text{CondSet}^{\text{light}} \) maps a profinite set \( S \) to its condensification \( \_S \). Since \( \text{Prof}^{\text{light}} \) is a small category, any condensed set can be written as a colimit of light profinite sets. More precisely, we have that

\[
T = \lim_{S \to T} \_S
\]

as a condensed set. From now on we will not make further distinction between \( S \) and \( \_S \) for \( S \) a light profinite set.
As we saw in the previous example, there is a natural functor from topological spaces to light condensed sets by mapping from light profinite sets. The following proposition shows that this functor is fully faithful in a reasonable subcategory of topological spaces (cf. [CS19, Proposition 1.7])

**Proposition 2.2.4.** The condensification functor has a left adjoint called the "underlying topological space", mapping a condensed set $T$ to the topological space given by

$$T(*)_\text{top} = \lim_{S \to T} S$$

where the colimit is taken in the category of topological spaces. More precisely, $T(*)_\text{top}$ has underlying set $T(*)$ and topology determined by the set of maps

$$\bigsqcup_{S \to T} S \to T(*)_\text{top}.$$

In particular, the functor $(-)$ is fully faithful in metrizably compactly generated spaces (e.g. metrizable compact Hausdorff spaces).

**Proof.** Since $T = \lim_{S \to T} S$ as a condensed set, the statement reduces to the fact that for a profinite set $S$ and a topological space $X$ we have

$$X(S) = C(S, X).$$

□

**Remark 2.2.5.** Let us make more explicit what means to be an epimorphism for topological spaces when considered as condensed sets. Let $X \to Y$ be a map of topological spaces such that their condensification $\overline{X} \to \overline{Y}$ is an epimorphism. This means that for any light profinite set $S$ and any map $f : S \to Y$, there is a surjection from a light profinite set $S' \to S$ and a map $S' \to X$ lifting $S$. For example, if $X \to Y$ is a surjection of compact Hausdorff spaces then so is its condensification. However, this property does not hold true for example in the case of a inductive limit $\lim_{n \to \infty} B_n$ of Banach spaces with injective transitions maps (LB spaces) in the case the maps are not of compact type (the closure of the image of a ball is compact), for the quotient map $\bigsqcup B_n \to \lim B_n$. In other words, the condensification of $\lim_{n \to \infty} B_n$ is not necessarily the colimit of the condensification of the Banach spaces $B_n$ unless the maps are compact.

In every topos there is a notion of quasi-compact and quasi-separated objects, in the case of light condensed abelian groups these properties can be stated in more concrete terms.

**Definition 2.2.6.** A condensed set $T$ is quasi-compact if there is a surjection $S \to T$ from a profinite set. A condensed set $T$ is quasi-separated if for every two maps from profinite sets $S \to T \leftarrow S'$, the fiber product $S \times_T S'$ is quasi-compact.

**Remark 2.2.7.** By definition, the Grothendieck topology of $\text{Prof}^{\text{light}}$ is finitary, this makes the profinite sets quasi-compact objects in the topos of condensed sets. Moreover, since light profinite sets are stable under countable limits, they are stable under pullbacks and so they are quasi-separated. This makes CondSet a coherent topos. On the other hand, if $T$ is a condensed set and $S, S' \to T$ are maps from profinite sets to $T$, then $S \times_T S'$ is a subobject of $S \times S$, therefore $T$ is quasi-separated if and only if for all $S, S'$ as before $S \times_T S'$ is also profinite.

We can describe concretely the qcqs objects in CondSet.

**Proposition 2.2.8.** Let $\text{CHaus}^{\text{light}}$ be the category of metrizable compact Hausdorff spaces. Then the condensification functor induces an equivalence from $\text{CHaus}^{\text{light}}$ to the category of qcqs condensed sets. Moreover, the category of quasi-separated condensed sets is equivalent to the ind-category with injective transition maps of metrizable compact Hausdorff spaces $\text{Ind}_{\text{inj}}(\text{CHaus}^{\text{light}})$. 


Proof. First, we claim that a quasi-compact subobject of a light profinite set is necessarily profinite. For this, let \( f : S \to S' \) be a map of light profinite sets, we want to see that the image of \( f \) is a closed subspace of \( S' \). Let \( \text{Im}(f) \subset S' \) be the image as topological space, it is profinite and we know that \( f \) factors through the condensification of \( \text{Im}(f) \). Then, we are left to show that if \( f \) is a surjection of light profinite sets then it is an epimorphism as condensed sets, but this is clear by the definition of the Grothendieck topology of \( \text{Prof}^{\text{light}} \).

Let \( T \) be a qcqs object in \( \text{CondSet} \), then there is a surjection \( S \to T \) from a light profinite set such that \( S \times_T S \) is also profinite. Then, \( T \) arises as the quotient of a light profinite set by a light profinite equivalence relation, making \( T(\ast)_{\text{top}} \) a metrizable compact Hausdorff space, the natural map \( T(\ast)_{\text{top}} \to T \) from the adjunction is an equivalence by Remark 2.2.5. Conversely, let \( X \) be a metrizable compact Hausdorff space and fix a countable basis \( \mathcal{U} \) of \( X \). Let \( I \) denote the countable cofiltered set of finite covers of \( X \) by 2 by \( 2 \) different elements in \( \mathcal{U} \), and for each \( i \in I \) let \( S_i = \{ U_{j_1}, \ldots, U_{j_{k_i}} \} \) be the cover of \( X \). Then \( S = \lim_i S_i \) is a light profinite set. We can define \( f : S \to X \) by mapping a system of open subsets \( x = \{ U_{j_i} \}_{i \in I} \) to its intersection \( f(x) = \bigcap_i U_{j_i} \), which is necessarily a point. The map \( f \) is then continuous and a surjection from a light profinite set onto \( X \). By Remark 2.2.5 the map of condensed sets \( S \to X \) is surjective, and the fiber product \( S \times_X S \) is the condensification of the topological fiber product which is a light profinite set, this shows that \( X \) is qcqs as wanted.

Finally, let \( T \) be a quasi-separated light condensed set, and let \( S \to T \) be a map from a profinite set \( S \). Then the image \( X \) of \( S \) in \( T \) is qcqs since \( S \times_X S = S \times_T S \) is profinite. This shows that \( T \) can be written as a union of qcqs condensed sets by injective maps, which produces an object in \( \text{Ind}_{\text{light}}(\text{CHaus}) \), furthermore, since qcqs condensed sets are compact objects in \( \text{CondSet} \) this map is fully faithful. Conversely, given a cofiltered diagram \( \{ X_i \} \) of light compact Hausdorff spaces with injective transition maps, the colimit \( T = \lim_i X_i \) of condensed sets is quasi-separated, namely, given any two maps from profinite sets \( S, S' \to T \) there is some \( i \) such that \( S, S' \) factor through \( X_i \), and \( S \times_T S' = S \times_{X_i} S' \) is profinite.

2.3. Light condensed abelian groups. Next, we define light condensed abelian groups and prove some of its most important features.

Definition 2.3.1. The category of light condensed abelian groups \( \text{CondAb}^{\text{light}} \) is the category of abelian group objects in \( \text{CondSet}^{\text{light}} \). Equivalently, it is the category of abelian sheaves on light profinite sets.

Example 2.3.2. (1) The forgetful functor

\[
\text{CondAb}^{\text{light}} \to \text{CondSet}
\]

has a left adjoint \( T \mapsto \mathbb{Z}[T] \) given by the free abelian group generated by a condensed set. The condensed abelian group \( \mathbb{Z}[T] \) is given by the sheafification of the functor mapping a light profinite set \( S \) to the free abelian group \( \mathbb{Z}[T(S)] \).

(2) Let \( A \) be a topological abelian group, then \( A \) has a natural structure of light condensed abelian group. Indeed, the condensation functor preserves finite limits and the structure of an abelian group for \( A \) is encoded in some diagrams such as \( + : A \times A \to A \).

(3) Let \( \mathbb{R} \) be the real numbers endowed with the addition and its natural topology, then \( \mathbb{R} \) is a condensed abelian group. On the other hand, if \( \mathbb{R}^\delta \) is endowed with the discrete topology then \( \mathbb{R}^\delta \) is another condensed abelian group with same underlying group as \( \mathbb{R} \). There is an inclusion \( \mathbb{R}^\delta \subset \mathbb{R} \) which is not an isomorphism. Indeed, for a light profinite set \( S \) we have

\[
\mathbb{R}/\mathbb{R}^\delta(S) = C(S, \mathbb{R})/C_{\text{lc}}(S, \mathbb{R}),
\]

where \( C_{\text{lc}}(S, \mathbb{R}) \) is the space of locally constant functions from \( S \) to \( \mathbb{R} \).
**Theorem 2.3.3.** The category CondAb\textsuperscript{light} is a Grothendieck abelian category endowed with a natural symmetric monoidal structure and an internal Hom. Moreover, it has the following properties.

1. Countable products are exact (countable AB\textsubscript{4\#}) and satisfy (AB\textsubscript{6}).
2. Sequential limits of surjective maps are surjective.
3. The object \( \mathbb{Z}[\mathbb{N} \cup \{\infty\}] \) is internally projective.

**Proof.** The fact that CondAb\textsuperscript{light} is a Grothendieck abelian category is a general fact about sheaves on abelian groups on a site. It also has a natural tensor product given by the sheafification of the tensor product of presheaves (in particular for \( A, B \in \text{CondAb}^{\text{light}} \) we have \( (A \otimes B)(\ast) = A(\ast) \otimes B(\ast) \)). The internal Hom is just the right adjoint of the tensor product. Point (1) follows from point (2) which is Remark 2.2.2. It is just left to prove point (3).

It suffices to prove that the space of null sequences \( P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}] / (\infty) \) is internally projective. We want to show that for a surjection \( A \to B \) of light condensed abelian groups, and that for all light profinite set \( S \), and a map \( g : \mathbb{Z}[S] \otimes N \to B \), there is a dashed arrow making the following diagram commutative

\[
\begin{array}{ccc}
\mathbb{Z}[S] \otimes P & \longrightarrow & A \\
\downarrow & & \downarrow \\
\mathbb{Z}[S] \otimes P & \xrightarrow{g} & B
\end{array}
\]

after possibly replacing \( S \) by a cover. We have that \( \mathbb{Z}[S] \otimes P = \mathbb{Z}[S \times (N \times \{\infty\})] / (\mathbb{Z}[S \times \{\infty\}]) \). Then the map \( g \) is the same as a map \( S \times (N \times \{\infty\}) \to B \) sending \( S \times \{\infty\} \) to 0. By hypothesis, there is a surjection \( f : S' \to S \times (N \cup \{\infty\}) \) and a map \( S' \to A \) lifting \( g \). For \( n \in \mathbb{N} \) let \( S'_n \) be the fiber over \( S \times \{n\} \) (which is still a surjection). By Proposition 2.1.6 we can find retractions \( r_n : S' \to S'_n \subset S' \), and construct the following diagram of locally profinite sets

\[
\begin{array}{ccc}
S' \times \mathbb{N} & \xrightarrow{\bigcup_n r_n} & S' \\
\downarrow & & \downarrow f \\
S \times (N \cup \infty) & \xrightarrow{\bigcup_n \text{for} n} & S \times (N \cup \{\infty\})
\end{array}
\]

We can find a light profinite compactification \( S'' \) of \( S' \times \mathbb{N} \) such that \( S \times \mathbb{N} \to S' \) extends to \( S'' \to S' \) (Exercise, construct one of such compactifications). Let \( D \) be the boundary of \( S'' \), by Proposition 2.1.6 we can find another retraction \( r : S'' \to D \). Let \( h : S'' \to S' \to A \) be the composite map, then \( h - h \circ r \) induces a map

\[
\mathbb{Z}[S''] / \mathbb{Z}[D] = \mathbb{Z}[S'] \otimes P \to A
\]

that lifts \( g \) proving what we wanted.

**Remark 2.3.4.** It is surprising that the object \( \mathbb{Z}[\mathbb{N} \cup \{\infty\}] \) is internally projective in CondAb\textsuperscript{light}. This does not happens at the level of profinite sets, for example the map \( (2\mathbb{N} \cup \{\infty\}) \sqcup (2\mathbb{N} + 1 \cup \{\infty\}) \to \mathbb{N} \cup \{\infty\} \) does not admit a split. This condensed abelian group will be key in the construction of examples on analytic rings.

We can define the condensed cohomology as follows:

**Definition 2.3.5.** Let \( T \in \text{CondSet}^{\text{light}} \) be a light condensed set and \( M \) a discrete abelian group, we define the condensed cohomology of \( T \) with values in \( M \) to be

\[
\text{RG}_{\text{cond}}(T, M) := \text{RHom}(\mathbb{Z}[T], M).
\]

Condensed cohomology behaves as expected in good cases.

**Proposition 2.3.6.** \([\text{CS19} \text{ Theorem } 3.2]\). Let \( S \) be a profinite set and \( M \) a discrete abelian group, then

\[
\text{RG}_{\text{cond}}(S, M) = C(S, M)
\]
is the space of continuous (eq. locally constant) functions from $S$ to $M$.

**Proof.** It is clear that $H^0_{\text{cond}}(S, M)$ is just the space of continuous maps from $S$ to $M$. To show that the higher cohomology groups vanish, it suffices to show that for a cover $S' \to S$ with Čech nerve $(S' \times_S S')_{[n] \in \Delta^op}$ the Čech cohomology complex

$$0 \to C(S', M) \to C(S' \times_S S', M) \to \cdots$$

(2.1)

is acyclic in cohomological degrees $\geq 1$. For this, we can write the surjection $S' \to S$ as a sequential limit of finite sets with surjective maps $\lim \leftarrow \n (S'_n \to S_n)$. Then the Čech complex (2.1) is the colimit of the Čech complexes of the surjections $S'_n \to S_n$, which are acyclic in degrees $\geq 1$ since any surjection of finite sets splits. □

**Proposition 2.3.7** ([CS19, Theorem 3.2]). Let $X$ be a light compact Hausdorff space and $M$ a discrete abelian group, then there is a natural isomorphism

$$R\Gamma_{\text{cond}}(X, M) = R\Gamma(X, M)$$

between condensed and Čech cohomology.

**Proof.** Since $X$ is compact Hausdorff we can formally reduce to the case $M = \mathbb{Z}$. Let $X_{\text{Prof}} := \text{Prof}^{\text{light}}_X$ be the site of light profinite sets over $X$. Then condensed cohomology of $X$ is the same as the cohomology in $X_{\text{Prof}}$. Let $X_{\text{top}}$ be the site consisting on closed subspaces of $X$ with coverings given by finite unions of closed subspaces admitting an open cover refinement. Then Čech cohomology of $X$ is the same as the cohomology on $X_{\text{top}}$. We have a natural morphism of sites

$$\eta : X_{\text{Prof}} \to X_{\text{top}}.$$

It suffices to show that the natural map $\mathbb{Z} \to R\eta_*\mathbb{Z}$ is an isomorphism. This can be proved at stalks, so let $x \in X$, then the stalk $R\eta_*\mathbb{Z}|_x$ is the same as the pushforward of the fiber over $x$, which is nothing but the condensed cohomology of a point which is $\mathbb{Z}$. □

3. Light solid abelian groups

The theory of solid abelian groups was introduced in [CS19], it plays a fundamental role in non-archimedean analytic geometries and non-archimedean analysis. The category Solid of solid abelian groups is a full subcategory of $\text{CondAb}$, stable under limits, colimits and extensions, and containing $\mathbb{Z}$; it is actually the smallest category satisfying those properties. In its "classical construction" the theory of locally compact abelian groups and its extensions as condensed abelian groups play a key role. However, within the new framework of light condensed mathematics, the theory of solid abelian groups can be formally developed from the more intuitive idea that the "summable sequences" in non-archimedean analysis are precisely the "null-sequences". In the following we will explain how this very simple idea naturally guides us to the correct definition of Solid.

3.1. Null-sequences and summability. Let $K$ be a local field and $V$ a Banach space over $K$. Recall that a null-sequence in $V$ is a sequence $(v_n)_{n \in \mathbb{N}}$ converging to 0. Similarly, a summable sequence is a sequence $(v_n)_{n \in \mathbb{N}}$ such that the partial sums $\sum_{i=0}^n v_i$ converge to an element in $v$ that we denote by $\sum_n v_n$. One of the first properties that we learn in a course of analysis is that a summable sequence $(v_n)$ has tails $u_n = \sum_{i \geq n} v_i$ converging to 0. In other words, we have a map

$$\{\text{summable sequences}\} \to \{\text{null sequences}\} : (v_n) \mapsto (u_n).$$

---

If we are allowed to call classical a construction just made around five-six years ago.
On the other hand, given a null sequence \( \{w_n\}_{n \in \mathbb{N}} \) we can form the sequence \( x_n := w_n - w_{n+1} \) which turns out to be summable in \( V \), namely,

\[
v_n := \sum_{i=0}^{n} x_n = w_0 - w_{n+1}
\]

and \( (v_n) \) converges to \( w_0 \) as \( n \to \infty \). Thus, we get a bijection

\[
\{ \text{null sequences} \} \to \{ \text{summable sequences} \} : (w_n)_{n} \mapsto (x_n)_{n} = (w_n - w_{n+1}).
\]

Nonetheless, any summable sequence in \( V \) is also a null-sequence. The converse does not hold for archimedean fields (e.g. \( (1/n) \)), but it does for non-archimedean fields thanks to the ultrametric inequality.

Therefore, a way to isolate non-archimedean analysis from condensed abelian groups is by asking that any null-sequence is summable, namely, that the map

\[
1 - S : \{ \text{null sequences} \} \to \{ \text{null sequences} \},
\]

where \( S \) is the shift map \( (v_n) \mapsto (v_{n+1}) \), is a bijection.

In order to formalize this idea, first we need to be able to talk about null-sequences of condensed abelian groups.

**Definition 3.1.1.** We let \( P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty) \). Given a condensed abelian group \( A \) its space of null sequences is given by \( \text{Null}(A) = \text{Hom}(P, A) \), we also let \( \text{Null}(A) := \text{Hom}(P, A) \).

**Example 3.1.2.** We continue in the spirit of Example 2.2.3 (1). For a quasi-separated condensed abelian group \( A \) being a null-sequence is an actual property of the underlying sequence, namely, the map

\[
\text{Null}(A) \to \text{Map}(\mathbb{N}, A) = \prod_{\mathbb{N}} A(*)
\]

is injective. However, for general condensed abelian groups null-sequences are not properties but additional structure you put in the condensed abelian group. As example, let \( \mathbb{R} \) be the real numbers with the usual topology, and let \( \mathbb{R}^\delta \) be the real numbers with the discrete topology. Then \( \mathbb{R}/\mathbb{R}^\delta \), if scary as topological abelian group, is a well defined condensed abelian group, and for any light profinite set \( S \) we have that

\[
\mathbb{R}/\mathbb{R}^\delta(S) = C(S, \mathbb{R})/C^{lc}(S, \mathbb{R})
\]

is the quotient of continuous maps from \( S \to \mathbb{R} \) modulo locally constant maps from \( S \to \mathbb{R} \). Applying this to \( S = \mathbb{N} \cup \{\infty\} \) we get that \( \mathbb{R}/\mathbb{R}^\delta(S) \) is a non-zero space of null-sequences while \( \mathbb{R}/\mathbb{R}^\delta(*) = 0 \), this shows that a null-sequence in that non quasi-separated quotient remembers the tails of the virtually zero sequence.

An additional feature for \( P \) is that it has a natural structure of algebra making \( \mathbb{Z}[T] = \mathbb{Z}[\mathbb{N}] \to \mathbb{P} \) an algebra morphism.

**Proposition 3.1.3.** The map addition map

\[
\mathbb{N} \times \mathbb{N} \to \mathbb{N}
\]

induces an algebra structure on \( P \), we shall denote this algebra by \( \mathbb{Z}[\hat{q}] \).

To prove Proposition 3.1.3 it will suffices to show the following lemma

**Lemma 3.1.4.** Consider a surjective map of light profinite sets \( S \to S' \) and let \( U \subset S' \) be an open subspace such that \( S \times_S U \to U \) is an homeomorphism. Let \( D \) and \( D' \) be the complements of \( U \) in
$S$ and $S'$ respectively. Then we have a pushout square in CondSet

\[
\begin{array}{ccc}
D & \longrightarrow & S \\
\downarrow & & \downarrow \\
D' & \longrightarrow & S'.
\end{array}
\]

Proof. We have a surjection of condensed sets $S \to S'$ whose Čech fiber is given by $S \times_{S'} S = \Delta S \cup D \times_{D'} D \subset S \times S$. Since $S \to S'$ is surjective, we have that $S' = S/(S \times_{S'} S) = S/(\Delta S \cup D \times_{D'} D)$, which is exactly the pushout $S \bigcup_D D'$.

Definition 3.1.5. Let $U$ be a light locally profinite set, i.e. a countable disjoint union of light profinite set. We let $P_U := \mathbb{Z}[U \cup \{\infty\}]/(\infty)$ be the space of "measures on $U$ vanishing at $\infty$".

Proposition 3.1.6. Let $U$ be a light locally profinite set, let $S$ be any compactification of $U$ and let $D$ be the boundary, then there is a natural isomorphism $P_U = \mathbb{Z}[S]/\mathbb{Z}[D]$.

Proof. We have a pushout diagram

\[
\begin{array}{ccc}
D & \longrightarrow & S \\
\downarrow & & \downarrow \\
* & \longrightarrow & U \cup \{\infty\},
\end{array}
\]

applying the left adjoint $\mathbb{Z}[\cdot]$ we get a push out diagram at the level of free modules, which induces the isomorphism

$$\mathbb{Z}[S]/\mathbb{Z}[D] = \mathbb{Z}[S \cup \{\infty\}]/(\infty).$$

□

Proof of Proposition 3.1.5. We can endow $\mathbb{N} \cup \{\infty\}$ with a structure of additive monoid by declaring $\infty + a = \infty$. Then, $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ has a natural algebra structure such that $\mathbb{Z}[\infty]$ is an ideal, this endows $P$ with an algebra structure. More explicitly, consider the addition map

$$(\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\}) \to \mathbb{N} \cup \{\infty\},$$

it sends the boundary of $\mathbb{N} \times \mathbb{N}$ to the boundary of $\mathbb{N}$, and by Proposition 3.1.6 it defines a map

$$P \otimes P \to P,$$

compatible with the multiplication map $\mathbb{Z}[T] \otimes \mathbb{Z}[T] \to \mathbb{Z}[T]$. It is easy to check that this defines an algebra structure on $P$.

□

3.2. Solid abelian groups form an analytic ring. Now we define the category of solid abelian groups, for this, note that the solid abelian group $P$ parametrizing null sequences has an endomorphism $\text{Shift} : P \to P$ which is induced from the map of profinite sets $\mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$ mapping $\infty$ to $\infty$ and $n$ to $n + 1$, we call $\text{Shift}$ the shift map.

Definition 3.2.1. Consider the map $1 - \text{Shift} : P \to P$. A light condensed abelian group $A$ is said solid if the natural map

$$\text{Hom}(P, A) \xrightarrow{1-\text{Shift}^\ast} \text{Hom}(P, A)$$

is an isomorphism. We let $\text{Solid} \subseteq \text{CondAb}^{\text{light}}$ denote the full subcategory of (light) solid abelian groups.

More generally, given $C \in \mathcal{D}(\text{CondAb}^{\text{light}})$ an object in the $(\infty)$-derived category of condensed abelian groups, we say that $C$ if solid if the natural map

$$R\text{Hom}(P, C) \xrightarrow{1-\text{Shift}^\ast} R\text{Hom}(P, C)$$

is an equivalence. We let $\mathcal{D}(\text{CondAb})^{\square} \subseteq \mathcal{D}(\text{CondAb}^{\text{light}})$ be the full subcategory of solid objects.
Remark 3.2.2. By Theorem 2.3.3 the object $P$ is internally projective in the category of light condensed abelian groups, in particular there is no difference between the derived or non derived Hom space $\text{Hom}(P, A)$. This shows that $\text{Solid} \subset \mathcal{D}(\text{CondAb})^\Box$.

The main theorem regarding the category of solid abelian groups is the following:

**Theorem 3.2.3.** The category $\text{Solid}$ is a Grothendieck abelian category stable under limits, colimits and extensions in $\text{CondAb}$. Furthermore, the following properties hold:

1. $\mathbb{Z} \in \text{Solid}$.
2. There is a left adjoint $(-)^\Box : \text{CondAb} \to \text{Solid}$ for the inclusion that we call the solidification functor.
3. There is a unique symmetric monoidal structure $\otimes^\Box$ on $\text{Solid}$ making $(-)^\Box$ symmetric monoidal.
4. $\mathbb{R}^\Box = 0$ (solid abelian groups kill the archimedean theory).

Moreover, $\mathcal{D}(\text{CondAb})^\Box$ is a presentable full subcategory of $\mathcal{D}(\text{CondAb})$ stable under limits and colimits, and the following properties are satisfied:

5. The inclusion $\mathcal{D}(\text{CondAb})^\Box \to \mathcal{D}(\text{CondAb})$ has a left adjoint $(-)^L\Box$.
6. An object $C \in \mathcal{D}(\text{CondAb})$ is solid if and only if $H^i(C) \in \text{Solid}$ for all $i \in \mathbb{Z}$, i.e. the natural $t$-structure on $\mathcal{D}(\text{CondAb})$ induces a $t$-structure on $\mathcal{D}(\text{CondAb})^\Box$.
7. For $C \in \mathcal{D}(\text{CondAb})^\Box$ and $M \in \mathcal{D}(\text{CondAb})$ we have $H\text{Hom}(M, C) \in \mathcal{D}(\text{CondAb})^\Box$.
8. The category $\mathcal{D}(\text{CondAb})^\Box$ has a unique symmetric monoidal structure $\otimes^L\Box$ making $(-)^L\Box$ symmetric monoidal.
9. The natural map $\mathcal{D}(\text{Solid}) \to \mathcal{D}(\text{CondAb})$ of derived categories is fully faithful, and has essential image $\mathcal{D}(\text{CondAb})^\Box$.
10. The functor $(-)^L\Box$ is the left derived functor of $(-)\Box$.
11. The functor $\otimes^L\Box$ is the left derived functor of $\otimes^\Box$.
12. For $S = \lim_n S_n$ a light profinite set there is a natural equivalence 

   $\mathbb{Z}^\Box[S] := (\mathbb{Z}[S])^L\Box \cong \lim_n \mathbb{Z}[S_n] \cong \prod_n \mathbb{Z}$.

   In particular, $\mathbb{Z}^\Box[S]$ is a compact projective solid abelian group, and if $S$ is infinite $\mathbb{Z}^\Box[S]$ is a compact projective generator of $\text{Solid}$.
13. For $I$ and $J$ countable sets we have 

   $\prod_I \mathbb{Z} \otimes^L\Box \prod_J \mathbb{Z} = \prod_{I \times J} \mathbb{Z}$.
14. The object $\prod_N \mathbb{Z}$ is flat in $\text{Solid}$.

In [CS19] a lot of effort is made in order to prove Theorem 3.2.3 and the only obvious property was point (12), this is because solid abelian groups were constructed by first defining the functor of measures $S \mapsto \mathbb{Z}^\Box[S]$. Furthermore, property (14) is not true in arbitrary solid abelian groups (counter example due to Effimov). It turns out that with Definition 3.2.1 most of the theorem is immediate.

**Proposition 3.2.4.** The category $\text{Solid}$ is a Grothendieck abelian category. Furthermore, points (1)-(8) hold. Moreover, property (12) implies (9) and (10), and property (13) implies (11).

**Proof.** Recall that the category Solid is defined as the full subcategory of condensed abelian groups $A$ such that the map $1 - \text{Shift}^*$ on $\text{Hom}(P, A)$ is an isomorphism. Since $P$ is internally projective, this condition is clearly stable under limits, colimits and extensions in CondAb, making Solid an abelian category. The same argument shows that $\mathcal{D}(\text{CondAb})^\Box$ is stable under limits and colimits in $\mathcal{D}(\text{CondAb})$. It is left to show that Solid and $\mathcal{D}(\text{CondAb})^\Box$ are presentable, for this, consider...
Q = cone(P → lim_{1→Shift} P), then an object C is solid if and only if \( R\text{Hom}(Q, C) = 0 \). Presentability then follows from [Lur09, Theorem 5.5.3.18].

(1) By Proposition 2.3.6 for all \( S \in \text{Prof} \) we have that \( R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) = C(S, \mathbb{Z}) \) is the space of locally constant functions. This implies that

\[
\text{Hom}(P, \mathbb{Z}) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}.
\]

Then, the action of \( 1 - \text{Shift}^* \) maps a sequence \((a_0, a_1, \ldots)\) to \((a_0 - a_1, a_1 - a_1, \ldots)\), which clearly has by inverse

\[
(b_0, b_1, b_2, \ldots) \mapsto \left( \sum_{i \geq 0} b_i, \sum_{i \geq 1} b_i, \ldots \right).
\]

since the sequences are eventually zero.

(2) and (5) The existence of the left adjoint follows from the adjoint functor theorem [Lur09, Corollary 5.5.2.9].

(3) and (8) It suffices to show that the kernel of the adjoints \((-)^{\mathcal{L}}\) and \((-)^{L\mathcal{L}}\) are tensor ideals in Solid and \( \mathcal{D}(\text{CondAb})^{\mathcal{L}} \) respectively. Let us just explain the proof for \((-)^{L\mathcal{L}}\). Let \( A \in \mathcal{D}(\text{CondAb}) \) be such that \( A^{L\mathcal{L}} = 0 \) and let \( M \in \mathcal{D}(\text{CondAb}) \). To prove that \( (M \otimes^L A)^{L\mathcal{L}} = 0 \) it suffices to show that for all \( B \in \mathcal{D}(\text{CondAb})^{\mathcal{L}} \) we have

\[
R\text{Hom}(A \otimes^L M, B) = 0,
\]

but we have that

\[
R\text{Hom}(A \otimes^L M, B) = R\text{Hom}(A, R\text{Hom}(M, B)),
\]

and \( R\text{Hom}(M, B) \) is solid by (7), proving that (3.1) vanishes.

(4) Since \( \mathbb{R} \) is an algebra and the functor \((-)^{L\mathcal{L}}\) is symmetric monoidal, it suffices to show that

\[
\pi_0(\mathbb{R}^{L\mathcal{L}}) = \mathbb{R}^{\mathcal{L}} = 0.
\]

Moreover, for this it suffices to show that the unit map \( \mathbb{Z} \to \mathbb{R}^{\mathcal{L}} \) is zero. For this, consider the null-sequence in \( \mathbb{R} \)

\[
(1, 1/2, 1/2, 1/4, 1/4, 1/4, 1/4, \ldots)
\]

defining a map \( f : P \to \mathbb{R} \). By definition of the solidification, there is an unique map \( g : P \to \mathbb{R}^{\mathcal{L}} \) making the following diagram commutative

\[
\begin{array}{ccc}
P & \xrightarrow{f} & \mathbb{R} \\
\downarrow{1-\text{Shift}} & & \downarrow \\
P & \xrightarrow{g} & \mathbb{R}^{\mathcal{L}}.
\end{array}
\]

Let \([0] : \mathbb{Z} \to P\) be the inclusion in the zero-th component, then \( g \circ [0] : \mathbb{Z} \to \mathbb{R}^{\mathcal{L}} \) defines an element \( x \) (virtually given by \( 1 + \frac{1}{2} + \frac{1}{2} + \cdots \)). We claim that \( x = 2 + x \), this would show that \( 2 = 0 \) and that \( \mathbb{R}^{\mathcal{L}} = 0 \) since \( 2 \) is a unit.

Consider the maps

\[
F : \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] : [n] \mapsto [2n + 1] + [2n + 2]
\]

\[
G : \mathbb{Z}[\mathbb{N}] \to \mathbb{Z}[\mathbb{N}] : [n] \mapsto [2n + 1].
\]
These maps naturally extend to endomorphisms of $P$. We claim that we have a commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{F} & P \\
\downarrow{1-S} & & \downarrow{1-S} \\
P & \xrightarrow{G} & P,
\end{array}
$$

namely, we have

$$(1-\text{Shift}) \circ F([n]) = (1-\text{Shift})([2n+1]+[2n+2]) = [2n+1]-[2n+2]+[2n+2]-[2n+3] = [2n+1]-[2n+3]$$

and

$$G \circ (1-\text{Shift})([n]) = G([n]-[n+1]) = [2n+1]-[2n+3].$$

On the other hand, we have that $f \circ F = f$, namely it is the sequence

$$((\frac{1}{2} + \frac{1}{2}), (\frac{1}{4} + \frac{1}{4}), (\frac{1}{4} + \frac{1}{4}), (\frac{1}{8} + \frac{1}{8}), (\frac{1}{8} + \frac{1}{8}), (\frac{1}{8} + \frac{1}{8}), \ldots) = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots).$$

By uniqueness of the lift $g : P \to \mathbb{R}^\square$, we must have $g \circ G = g$. Then, if $g$ represents the null sequence $(x_0, x_1, x_2, x_3, \ldots)$, we must have $x_n = x_{2n+1}$ for all $n \in \mathbb{Z}$. In particular, $x_0 = x_1$, so that

$$0 = x_0 - x_1 = 1,$$

proving what we wanted.

(6) This follows from the fact that for all $C \in \mathscr{D}(\text{CondAb})$ we have

$$H^i (\text{RHom}(P, C)) = \text{Hom}(P, H^i (C))$$

for $i \in \mathbb{Z}$.

(7) Let $M \in \mathcal{D}(\text{CondAb})$ and $C \in \mathcal{D}(\text{CondAb})^\square$, then the claim follows from the isomorphism

$$\text{RHom}(P, \text{RHom}(M, C)) = \text{RHom}(M, \text{RHom}(P, B)),$$

and the fact that $B$ is solid.

Now let us assume that properties (11) and (12) hold.

(9) The map $\text{Solid} \to \text{CondAb}$ induces a functor of derived categories $\mathcal{D}(\text{Solid}) \to \mathcal{D}(\text{CondAb})$, by [Lur17] Proposition 1.3.3.7, and since $P^\square = \prod_n \mathbb{Z}$ is a compact projective generator of $\text{Solid}$, it suffices to show that for $A \in \text{Solid}$ we have

$$\text{RHom}(P^\square, A) = \text{Hom}(P^\square, A).$$

But we know that $P^\square = P^{L\square}$, and by the left adjoints of (2) and (5) we have

$$\text{RHom}(P^\square, A) = \text{RHom}(P^{L\square}, A) = \text{RHom}(P, A) = \text{Hom}(P, A) = \text{Hom}(P^\square, A).$$

(10) This follows from the fact that $\mathbb{Z}[S]^{L\square} = \mathbb{Z}[S]^\square$ sits in degree zero. Indeed, since both derived categories are right complete, it suffices to show that the restriction of $(-)^{L\square}$ to connective complexes $\mathcal{D}_{\geq 0}(\text{CondAb})$ (i.e. non-negative homological degrees) is the left derived functor. This statement boils to the fact that $(-)^{L\square} : \mathcal{D}_{\geq 0}(\text{CondAb}) \to \mathcal{D}_{\geq 0}(\text{Solid})$ is the left Kan extension of its restriction to the full subcategory of generators $\mathcal{C}^0 = \{\mathbb{Z}[S]_{S \in \text{Proflight}} \subset \mathcal{D}_{\geq 0}(\text{CondAb})\}$[2]. In other words, that for $C \in \mathcal{D}_{\geq 0}(\text{CondAb})$ we have

$$C^{L\square} = \lim_{\mathbb{Z}[S] \in \mathcal{C}^0/C} \mathbb{Z}[S]^{L\square} = \lim_{\mathbb{Z}[S] \in \mathcal{C}^0/C} \mathbb{Z}[S]^{\square}. \quad \text{(10)}$$

\[ \text{Note that the full subcategory } \mathcal{C}^0 \subset \mathcal{D}_{\geq 0}(\text{CondAb}) \text{ is not a full subcategory of } \text{CondAb} \text{ since the objects of } \mathcal{C}^0 \text{ are not projective} \]
(11) Finally, to show that \( \otimes^L \) is the left derived functor of \( \otimes \), it suffices to show that there is a family of compact projective generators \( \mathcal{C}^0 \subset \text{Solid} \) stable under the solid tensor product, such that for \( A, B \in \mathcal{C}^0 \) we have \( A \otimes^L B = A \otimes B \). Taking \( \mathcal{C}^0 \) as the full subcategory spanned by \( \mathbb{Z} \) with \( S \) light profinite we are done thanks to property (13).

**Corollary 3.2.5.** Let \( C \) be a real condensed vector space. Then \( C^L = 0 \).

**Proof.** The solidification functor \( (-)^L \) is symmetric monoidal, in particular \( \mathbb{R}^L \) is an algebra and \( C^L \) has a natural \( \mathbb{R}^L \)-module structure. But \( \mathbb{R}^L = 0 \), which implies that \( C^L = 0 \).

We have proven most of Theorem 3.2.3 it is left to show points (12)-(14) regarding the explicit description of the free objects \( \mathbb{Z}^S := \mathbb{Z}[S]^L \), their solid tensor products, and the flatness of \( \coprod_n \mathbb{Z} \) in Solid, we left those properties for the next sections.

### 3.3. Computing measures in solid abelian groups.

The objective in this section is to prove the following theorem

**Theorem 3.3.1.** Let \( S = \lim_n S_n \) be a light profinite set. Then the natural map of solid abelian complexes

\[
\mathbb{Z}[S]^L \to \lim_n \mathbb{Z}[S_n]
\]

is an equivalence. Furthermore, the following hold:

1. \( \prod_n \mathbb{Z} \) is a compact projective generator of Solid
2. The natural map \( \mathcal{D}(\text{Solid}) \to \mathcal{D}(\text{CondAb})^L \) is an equivalence of \( \infty \)-categories.
3. The functor \( (-)^L \) is the left derived functor of \( (-)^L \).

By Proposition 3.2.4 it is only left to prove the first assertion of the theorem, this will require some lemmas. Recall that \( P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty) \) is the solid abelian group parametrizing null-sequences.

First, we see that it suffices to compute the solidification of \( P \) in order to compute the solidification of \( \mathbb{Z}[S] \) for \( S \) a light profinite set.

**Lemma 3.3.2.** Let \( S \) be a light profinite set, there is a map \( P \to \mathbb{Z}[S] \) that induces isomorphisms on solidifications

\[
P^L \cong \mathbb{Z}[S]^L.
\]

**Proof.** Let us write \( S = \lim_n S_n \) as a limit of finite sets with surjective transition maps and projections \( \pi_n : S \to S_n \). We can find a sequence of compatible lifts \( S_0 \to S_1 \to S_2 \to \cdots \to S \) with \( \iota_n : S_n \to S \). Enumerating \( \bigcup_n \iota_n(S_n) \cong \mathbb{N} \) along the previous inclusions, we get an injection \( \mathbb{N} \to S \). Then for \( a \in \iota_n(S_n) \cap \iota_{n-1}(S_{n-1}) \subset \mathbb{N} \) consider the element \( \iota_n(a) - \iota_{n-1}(a) \). The sequence \( \{\iota_n(a) - \iota_{n-1}(a)\}_{n \in \mathbb{N}} \) converges to zero in \( \mathbb{Z}[S] \) and defines an injective map \( g : P \to \mathbb{Z}[S] \). We claim that \( g \) induces an isomorphism after solidification.

We claim that we have a commutative diagram

\[
\begin{array}{ccc}
P \otimes \mathbb{Z}[S] & \xrightarrow{F} & \mathbb{Z}[S] \\
\downarrow^{(1-\text{Shift}) \otimes \text{id}_S} & & \downarrow \\
P \otimes \mathbb{Z}[S] & \xrightarrow{G} & P
\end{array}
\]

where the top horizontal arrow \( F \) arises from a map \( (\mathbb{N} \times \{\infty\}) \times S \to \mathbb{Z}[S] \) that vanishes at \( \infty \times S \). This map is given by the sequence of maps \( \{n\} \times S \to \mathbb{Z}[S] \) given by \( \text{id}_{\mathbb{Z}[S]} \) if \( n = 0 \) and \( \text{id}_{\mathbb{Z}[S]} - \iota_{n-1} \circ \pi_{n-1} \) if \( n \geq 1 \), which vanish uniformly on \( S \) at \( \infty \). Then, to define the lower horizontal arrow \( G \) we need to show that the composite \( F \circ (1-\text{Shift}) \) lands in \( P \), but the composite corresponds to the map of condensed sets

\[
G : (\mathbb{N} \cup \{\infty\}) \times S \to \mathbb{Z}[S]
\]
vanishing at $\infty \times S$, and given by $\iota_{n-1} \circ \pi_{n-1} - \iota_n \circ \pi_n : S \to \mathbb{Z}[S]$ on $\{n\} \times S$ (where we make the convention $\iota_{-1} \circ \pi_{-1} = 0$). In particular, $G(\{n\} \times S)$ lands in $P$, and so it extends to a map $G : (\mathbb{N} \cup \{\infty\}) \times S \to P$ that vanishes at $\{\infty\} \times S$, producing the desired factorization.

Taking solidifications of (3.2), we get a commutative diagram

$$
\begin{array}{ccc}
(P \otimes \mathbb{Z}[S])^L & \xrightarrow{F} & \mathbb{Z}[S]^L \\
\uparrow & & \uparrow \\
(P \otimes \mathbb{Z}[S])^L & \xrightarrow{G} & P^L
\end{array}
$$

where the left vertical arrow is an isomorphism, and the top horizontal arrow has a section induced from the map $\{0\} \times S \to P \otimes \mathbb{Z}[S]$. The previous shows that $\mathbb{Z}[S]^L$ is a retract of $P^L$ with idempotent morphism $r : P^L \to P^L$. To show that the map is an actual isomorphism we need to show that $r$ is the identity. To prove this last claim, note that the diagram (3.2) restricts to a diagram

$$
\begin{array}{ccc}
P \otimes P & \xrightarrow{F} & P \\
(1-\text{Shift}) \otimes \text{id}_P & \uparrow & \text{id}_P \uparrow \\
P \otimes P & \xrightarrow{\text{id}_P} & P
\end{array}
$$

via the inclusion $P \subset \mathbb{Z}[S]$. Indeed, the map $F$ is given by the sequence of endomorphisms $\text{id}_{\mathbb{Z}[S]} - \iota_{n-1} \circ \pi_{n-1}$ of $\mathbb{Z}[S]$, which restrict to the endomorphisms $\text{id}_P - \iota_{n-1} \circ \pi_{n-1}$ of $P$. Taking solidifications we get

$$
\begin{array}{ccc}
(P \otimes P)^L & \xrightarrow{F} & P^L \\
\uparrow & & \uparrow \\
(P \otimes P)^L & \xrightarrow{\text{id}_P} & P^L
\end{array}
$$

and the idempotent $r$ obtained from (3.3) is the same as the idempotent obtained from (3.4) which is the identity.

Now, we compute the solidification of $P$. We apply the same trick as in the proof of Lemma 3.3.2 to replace $P$ by a simpler condensed abelian group.

**Lemma 3.3.3.** Let $\prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z} = \bigcup_{n \in \mathbb{N}} \prod_{n} \mathbb{Z} \cap [-n, n] \subset \prod_{n} \mathbb{Z}$ be the condensed set of bounded sequences of integers. Consider the natural map $P \to \prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z}$ induced by the null sequence $e_n \in \prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z}$ with $e_n = (0, 0, \cdots, 0, 1, 0, \cdots)$, which zero except for a 1 in the $n$-th component. Then the natural map $P^L \to \left(\prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z}\right)^L$ is an isomorphism.

**Proof.** We claim that there is a commutative square

$$
\begin{array}{ccc}
P \otimes \prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z} & \xrightarrow{F} & \prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z} \\
(1-\text{Shift}) \otimes \text{id}_P & \uparrow & \uparrow \\
P \otimes \prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z} & \xrightarrow{G} & P
\end{array}
$$

where the top horizontal arrow $F$ is given by the null-sequence of endomorphisms of $\prod_{n \in \mathbb{N}}^{\text{bnd}} \mathbb{Z}$ given by the projection $\pi_{\geq n}$ in the $\geq n$-components. To prove the claim, we need to see that the map
$G = F \circ (1 - \text{Shift})$ lands in $P$, but it is given by the null-sequence of endomorphisms of $\prod_{\mathbb{N}}^\text{bnd} \mathbb{Z}$ given by the projections $\pi_n = \pi_{\geq n} - \pi_{\geq n+1}$, whose target is in $P$. Taking solidifications of (3.5) we get a commutative diagram

\[
\begin{array}{ccc}
(P \otimes \prod_{\mathbb{N}}^\text{bnd} \mathbb{Z})^{L\square} & \xrightarrow{F} & (\prod_{\mathbb{N}}^\text{bnd} \mathbb{Z})^{L\square} \\
\uparrow & & \uparrow \\
(P \otimes \prod_{\mathbb{N}}^\text{bnd} \mathbb{Z})^{L\square} & \xrightarrow{G} & P^{L\square}
\end{array}
\]

such that the top horizontal arrow has a section given by the embedding in the 0-th component of the tensor. Then, as in the proof of Lemma 3.3.2, one gets an idempotent endomorphism $r : P^{L\square} \to P^{L\square}$ whose retract is $\mathbb{Z}[S]^{L\square}$, and to see that $r$ is the identity, it suffices to notice that (3.5) restricts to a commutative diagram of the form (3.4), and then one applies the argument as in the proof of Lemma 3.3.2.

\[\square\]

**Lemma 3.3.4.** The natural map $\prod_{\mathbb{N}}^\text{bnd} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$ induces an isomorphism in solidifications $\prod_{\mathbb{N}}^\text{bnd} \mathbb{Z}/\prod_{\mathbb{N}} \mathbb{Z} = \prod_{\mathbb{N}} \mathbb{Z}$.

**Proof.** Let $\prod_{\mathbb{N}}^\text{bnd} \mathbb{R} = \bigcup_n \prod_{\mathbb{N}} \mathbb{R} \cap [-n, n]$ be the condensed real vector space. We have isomorphisms of condensed abelian groups

\[\prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}} \mathbb{Z} = \prod_{\mathbb{N}} \mathbb{R}/\prod_{\mathbb{N}} \mathbb{R}.
\]

Indeed, this follows from the fact that we have short exact sequences

\[0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{R} \to \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z} \to 0
\]

and

\[0 \to \prod_{\mathbb{N}} \mathbb{Z}^{\text{bnd}} \to \prod_{\mathbb{N}} \mathbb{R}^{\text{bnd}} \to \prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z} \to 0.
\]

In particular, the quotient $\prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}}^\text{bnd} \mathbb{Z}$ can be endowed with an structure of $\mathbb{R}$-condensed vector space, and so its solidification vanishes by Corollary 3.2.5. This shows that

\[\prod_{\mathbb{N}}\mathbb{Z}^{L\square} = \prod_{\mathbb{N}}\mathbb{Z}
\]

as wanted. \(\square\)

**Corollary 3.3.5.** Let $S = \varprojlim_{n} S_n$ be a light profinite set, then we have natural isomorphisms

\[\mathbb{Z}^{\square}[S] = R\text{Hom}(C(S, \mathbb{Z}), \mathbb{Z})
\]

and

\[C(S, \mathbb{Z}) = R\text{Hom}(\mathbb{Z}^{\square}[S], \mathbb{Z}).
\]

**Proof.** The first isomorphism follows from the fact that $C(S, \mathbb{Z}) = \varprojlim_n C(S_n, \mathbb{Z})$ and that $\mathbb{Z}^{\square}[S] = \varprojlim_n \mathbb{Z}[S_n]$. The second isomorphism follows from the left adjoint $(-)^{L\square}$

\[R\text{Hom}(\mathbb{Z}^{\square}[S], \mathbb{Z}) = R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) = C(S, \mathbb{Z}).\]

\(\square\)
Corollary 3.3.6. Theorem [3.3.1] holds. Moreover, we have \( \prod_n \mathbb{Z} \otimes_{\mathbb{L}} \prod_n \mathbb{Z} = \prod_{n \times n} \mathbb{Z} \). In particular, \( \otimes_{\mathbb{L}} \) is the left derived functor of \( \otimes_\mathbb{C} \).

Proof. The consequences (1)-(3) of the theorem were proven in Proposition [3.2.4]. By Lemmas [3.3.2, 3.3.3] and [3.3.4] we know that \( \mathbb{Z}[S]^{L} \cong \prod_n \mathbb{Z} \) abstractly as solid abelian groups. Following the explicit isomorphisms constructed in the lemmas, one can verify that the previous isomorphism actually identifies with the natural arrow

\[
\mathbb{Z}[S]^{L} \cong \lim_n \mathbb{Z}[S_n].
\] (3.6)

More explicitly, this holds true for \( P \) by the proof of Lemmas [3.3.3 and 3.3.4]. In particular, we have natural isomorphisms \( R\text{Hom}(\prod_n \mathbb{Z}, \mathbb{Z}) = \bigoplus_n \mathbb{Z} \) and \( R\text{Hom}(\bigoplus_n \mathbb{Z}, \mathbb{Z}) = \prod_n \mathbb{Z} \). This shows that the objects \( \mathbb{Z}[S]^{L} \) are reflexive over \( \mathbb{Z} \), and it suffices to show that the map (3.6) becomes an isomorphism after taking duals. This follows from the fact that

\[
R\text{Hom}(\mathbb{Z}[S]^{L}, \mathbb{Z}) = R\text{Hom}(\mathbb{Z}[S], \mathbb{Z}) = C(S, \mathbb{Z})
\]

\[
= \lim_i C(S_i, \mathbb{Z}) = \lim_i R\text{Hom}(\mathbb{Z}[S_i], \mathbb{Z}) = R\text{Hom}(\lim_i \mathbb{Z}[S_i], \mathbb{Z}),
\]

where in the last equality we use that \( \lim_i \mathbb{Z}[S_i] \) is isomorphic to \( \prod_n \mathbb{Z} \) by Proposition [2.1.7].

On the other hand, we have an isomorphism \( P \times P \cong P \) given by taking an anti-diagonal enumeration of \( N \times N \). This shows that

\[
\prod_n \mathbb{Z} \otimes_{\mathbb{L}} \prod_n \mathbb{Z} \cong (P \otimes P)^{L} \cong P^{L} \cong \prod_n \mathbb{Z}.
\] (3.7)

An explicit description of this enumeration shows that the isomorphism (3.7) is given by the natural map

\[
\prod_n \mathbb{Z} \otimes_{\mathbb{L}} \prod_n \mathbb{Z} \cong \prod_n \mathbb{Z}.
\]

A first interesting property of the solidification functor is that it computes singular cohomology of CW complexes.

Proposition 3.3.7. Let \( X \) be a CW complex, then \( \mathbb{Z}[X]^{L} \) is equivalent to the complex of singular chains in \( X \).

Proof. Writing \( X \) as a colimit of finite CW complexes it suffices to construct a natural quasi-isomorphism between \( \mathbb{Z}[X]^{L} \) and the chain complex of \( X \), we can then assume \( X \) to be compact. Let \( S \to X \) be a surjection from a light profinite set with Čech nerve \( S_\bullet \to X \). We have a resolution

\[
\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[X] \to 0
\]

proving that \( \mathbb{Z}[X]^{L} \) is given by the connective complex.

\[
\cdots \to \mathbb{Z}[S_2] \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to 0.
\]

By Corollary [3.3.5] the complex \( \mathbb{Z}[X]^{L} \) is reflexive, and to naturally identify it with singular chains it suffices to naturally identify its dual with singular cochains. But

\[
R\text{Hom}(\mathbb{Z}[X]^{L}, \mathbb{Z}) = R\text{Hom}(\mathbb{Z}[X], \mathbb{Z}) = R\Gamma_{\text{cond}}(X, \mathbb{Z})
\]

is the condensed cohomology of \( X \), that we identified with sheaf cohomology on \( X \) by Proposition [2.3.7] and so with singular cochains. \(\square\)
3.4. Flatness of $\prod_N \mathbb{Z}$ and the structure of Solid. In this section we prove the last part of Theorem 3.2.3 regarding the flatness of $\prod_N \mathbb{Z}$ as solid abelian group. The proof strategy begins by first describing all the finitely presented solid abelian groups.

**Definition 3.4.1.** A solid abelian group is said finitely generated if it is a quotient of $\prod_N \mathbb{Z}$. A solid abelian group is said finitely presented if it is a cokernel of a map $\prod_N \mathbb{Z} \to \prod_N \mathbb{Z}$. 

**Theorem 3.4.2.** The finitely presented objects of Solid form an abelian category stable under kernels, cokernels and extensions in Solid, such that Solid = Ind(Solid$_{\text{finpres}}$). Moreover, any $M \in$ Solid$_{\text{finpres}}$ has a resolution

$$0 \to \prod_N \mathbb{Z} \to \prod_N \mathbb{Z} \to M \to 0.$$ 

A first corollary is the flatness of $\prod_N \mathbb{Z}$.

**Corollary 3.4.3.** The solid abelian group $\prod_N \mathbb{Z}$ is flat for the solid tensor product.

**Proof.** Since Solid = lim(Null(Solid$_{\text{finpres}}$)), it suffices to show that for $M$ a finitely presented solid abelian group $M \otimes L \square \prod_N \mathbb{Z}$ sits in degree 0. By the Theorem 3.4.2 we have a resolution

$$0 \to \prod_N \mathbb{Z} \to \prod_N \mathbb{Z} \to M \to 0.$$ 

Tensoring with $\prod_N \mathbb{Z}$, and using Corollary 3.3.6 we see that $M \otimes L \square \prod_N \mathbb{Z} = \prod_N M$ which clearly sits in degree 0. ∎

In order to proof Theorem 3.4.2 we shall need the following lemma.

**Lemma 3.4.4.** Any finitely generated submodule of $\prod_N \mathbb{Z}$ is isomorphic to $\prod_I \mathbb{Z}$ with $I$ countable.

**Proof.** Let $M \subset \prod_N \mathbb{Z}$ be a finitely generated subobject, then $M$ is the image of a map $f : \prod_N \mathbb{Z} \to \prod_N \mathbb{Z}$, which is the dual of a map

$$g : \bigoplus_N \mathbb{Z} \to \bigoplus_N \mathbb{Z}. \quad (3.8)$$

We shall need the following claim:

**Claim.** Let $N$ be a countable abelian group that embeds in a direct product of $\mathbb{Z}$, then $N$ is free.

**Proof of the claim.** Let us pick a basis $\{e_n\}_{n \in \mathbb{N}}$ of $\mathbb{Q} \otimes N$, and let $N_n = (e_0, \ldots, e_n)_{\mathbb{Q} \cap N}$. It suffices to show that each $N_n$ is finite free, namely, we have $N = \bigcup_n N_n$ and the quotient $N_{n+1}/N_n$ is torsion free. We can assume without loss of generality that $\{e_n\}_{n \in \mathbb{N}} \subset N$. Then, it suffices to prove that $M_n = N_n/\langle e_1, \ldots, e_n \rangle\mathbb{Z}$ is finite. Suppose it is not, then we can find elements $x_m \in M_n$ of exactly $b_m$ torsion for $m \in \mathbb{N}$, so that $b_m \to \infty$ as $m \to \infty$. Taking lifts $y_m \in N_n$ of $x_n$ this implies that $y_m = \sum_{i=0}^n c_{i,m} e_i$ with coefficients satisfying the following properties:

- $c_{i,m} = 0$ or $\text{GCD}(c_{i,m}d_{i,m}) = 1$,
- $\text{lcm}(d_{i,m}) = b_m$.

By hypothesis $N$ embeds into $\prod_I \mathbb{Z}$. Then, there is some projection $\prod_I \mathbb{Z} \to \prod_{J \subset I} \mathbb{Z}$ with $J$ finite such that the image of the elements $\{e_1, \ldots, e_n\}$ are linearly independent, proving that for $m >> 0$ the element $y_m$ cannot be mapped into $\prod_{i=0}^k \mathbb{Z}$ as $b_m \to \infty$ as $m \to \infty$, which is a contradiction. This proves the claim. ∎
We can decompose the map $g = j \circ h$ in (3.8) as a split surjection $h : \bigoplus_{\mathbb{Z}} \mathbb{Z} \to M$ and an injection $j : M \to \bigoplus_{\mathbb{N}} \mathbb{Z}$. We can then write short exact sequences

$$0 \to K \to \bigoplus_{\mathbb{N}} \mathbb{Z} \xrightarrow{h} M \to 0$$

and

$$0 \to M \to \bigoplus_{\mathbb{N}} \mathbb{Z} \to Q \to 0$$

with $M$ and $K$ free abelian groups. Taking duals we get exact sequences

$$0 \to M^\vee \to \prod_{\mathbb{N}} \mathbb{Z} \to K^\vee \to 0$$

and

$$0 \to \text{Hom}(Q, \mathbb{Q}) \to \prod_{\mathbb{N}} \mathbb{Z} \to M^\vee \to \text{Ext}^1(Q, \mathbb{Z}) \to 0.$$

Then, the composite

$$\prod_{\mathbb{N}} \mathbb{Z} \xrightarrow{f} \prod_{\mathbb{N}} \mathbb{Z} \to K^\vee$$

is zero and we can assume without loss of generality that $K = 0$ and so $g$ is injective. Thus, we have an exact sequence

$$0 \to \bigoplus_{\mathbb{N}} \mathbb{Z} \xrightarrow{g} \bigoplus_{\mathbb{N}} \mathbb{Z} \to Q \to 0.$$  \hfill (3.9)

Consider the natural map

$$Q \to \prod_{\text{Hom}(Q, \mathbb{Z})} \mathbb{Z}$$

and let $Q$ be its image. By the previous claim $Q$ is a free abelian group, and so $Q \to Q$ is a split surjection. Thus, by taking out the free direct summand, we can assume without of generality that $\text{Hom}(Q, \mathbb{Z}) = 0$. Then, one actually has that $\text{Hom}(Q, \mathbb{Z}) = 0$, namely, the $S$-valued points of the $\text{Hom}$ space are equal to $\text{Hom}(Q, C(S, \mathbb{Z}))$ and $C(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module. We deduce that the dual of (3.9) is the short exact sequence

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to \text{Ext}^1(Q, \mathbb{Z}) \to 0,$$

going that the image of $f$ is $\prod_{\mathbb{N}} \mathbb{Z}$ as wanted. \hfill \Box

**Proof of Theorem 3.4.2.** By the proof of Lemma 3.4.4 any finitely presented module $M \in \text{Solid}$ is of the form $M = \prod_{\mathbb{I}} \mathbb{Z} \oplus \text{Ext}^1(Q, \mathbb{Z})$ with $\mathbb{I}$ a countable set, and $Q$ a countable abelian group such that $\text{Hom}(Q, \mathbb{Z}) = 0$. By taking duals of a free resolution

$$0 \to \bigoplus_{\mathbb{N}} \mathbb{Z} \to \bigoplus_{\mathbb{N}} \mathbb{Z} \to Q \to 0,$$

we get a presentation

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \oplus \prod_{\mathbb{I}} \mathbb{Z} \to M \to 0$$

proving the second statement of the theorem. The stability of finitely presented solid modules under kernels, cokernels and extensions is then a standard fact for abelian categories for which finitely presented objects admit a resolution by compact projective generators (i.e. are pseudo-coherent, cf. [Sta22, Tag 064N] for the case of modules over rings). \hfill \Box
3.5. Examples of solid tensor products. We finish the discussion of solid abelian groups with some computations of solid tensor products that appear a lot in practice.

Example 3.5.1 (Power series ring). Let $\mathbb{Z}[[q]]$ be the ring of power series in one variable seen as a condensed ring. It is a solid abelian group since $\mathbb{Z}[[q]] = \lim_{\leftarrow n} \mathbb{Z}[q]/q^n$ is a limit of discrete modules. Indeed, if $\mathbb{Z}[q] = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/(\infty)$ is the algebra of null-sequences, see Proposition 3.1.3 we have $\mathbb{Z}[q]^{\square} = \mathbb{Z}[[q]]$. Corollary 3.3.6 implies that

$$\mathbb{Z}[[q_1]] \otimes^L \mathbb{Z}[[q_2]] = \mathbb{Z}[[q_1, q_2]].$$

On the other hand, the morphism of algebras $\mathbb{Z}[q] \to \mathbb{Z}[[q]]$ is idempotent when seen as solid algebras, namely,

$$\mathbb{Z}[[q]] \otimes^L \mathbb{Z}[[q]] = (\mathbb{Z}[[q_1]] \otimes^L \mathbb{Z}[[q_2]]) \otimes^L \mathbb{Z}[[q_1, q_2]] \mathbb{Z} = \mathbb{Z}[[q_1, q_2]] \otimes^L \mathbb{Z}[[q_1, q_2]] \mathbb{Z} = \mathbb{Z}[[q_1, q_2]]/\mathbb{Z}(q_1 - q_2) = \mathbb{Z}[[q]],$$

where $\mathbb{Z}[[q_1, q_2]]/\mathbb{Z}(q_1 - q_2)$ is the derived quotient, represented by a Koszul complex.

Example 3.5.2 ($p$-adic integers). The $p$-adic integers $\mathbb{Z}_p = \lim_{\leftarrow n} \mathbb{Z}/p^n$ is a solid abelian group being a limit of discrete abelian groups. We have a short exact sequence of solid abelian groups

$$0 \to \mathbb{Z}[[X]] \xrightarrow{X \mapsto 0} \mathbb{Z}[[X]] \to \mathbb{Z}_p \to 0,$$

indeed, this is the limit of the short exact sequences

$$0 \to \mathbb{Z}[X]/X^n \xrightarrow{X \mapsto 0} \mathbb{Z}[X]/X^n \to \mathbb{Z}/p^n \to 0.$$

Thus, the tensor $\mathbb{Z}_p \otimes^L \mathbb{Z}[[Y]]$ is nothing but $\mathbb{Z}_p[[Y]]$.

On the other hand, the tensor product $\mathbb{Z}_p \otimes^L \mathbb{Z}_\ell$ is represented by the complex

$$\mathbb{Z}_p[[X]] \xrightarrow{X \mapsto \ell} \mathbb{Z}_p[[X]],$$

if $\ell \neq p$ then $\mathbb{Z}_p \otimes^L \mathbb{Z}_\ell$ while if $\ell = p$ we get $\mathbb{Z}_p \otimes^L \mathbb{Z}_p = \mathbb{Z}_p$. In particular, $\mathbb{Z}_p$ is an idempotent $\mathbb{Z}$-algebra for the solid tensor product. In other words, being a $\mathbb{Z}_p$-module is not additional structure but a property for solid abelian groups!

Example 3.5.3 ($I$-adically complete modules). Given a discrete ring $A$ and $I$ a finitely generated ideal, there is a notion of being derived $I$-adically complete (see [Man22, Definition 2.12.3] and [Sta22, Tag 091N]). When $I = (a)$ is generated by a single element, and $A \xrightarrow{a} A$ is the multiplication by $a$, for an object $C$ in the derived category of (condensed) $A$-modules being $I$-adically complete is equivalent to the vanishing of the limit $R\lim_{\leftarrow n} C = 0$ given by multiplication along the complex $A \xrightarrow{a} A$. If we write $J \to A$ for $A \xrightarrow{a} A$, we can think of $J$ as a generalized Cartier divisor, namely, an invertible $A$-module together with a map $J \to A$. We can define powers of $J$ by tensoring, obtaining generalized Cartier divisors $J^n \to A$. Then, a $A$-modules $C$ is derived $I$-adically complete if the natural map

$$C \to R\lim_{\leftarrow n} C/I^n J,$$

where the quotient $C/I^n J$ is the pushout of $C$ along the map of derived rings $A \to A/I^n J$, where $A/I^n J$ is the dg-algebra given by the Koszul complex $J \to A$.

By [Man22, Lemma 2.12.9] if $A$ is a finitely generated $\mathbb{Z}$-algebra and $N, M$ are connective derived $I$-adically complete modules, then $N \otimes_{A, \square}^L M$ is also derived $I$-adically complete (here the tensor product is the natural one attached for a commutative ring object in Solid, equivalently, it is the solidification of the condensed tensor product over $A$).

Example 3.5.4 (Tensor product of $\mathbb{Q}_p$-Banach spaces). Specializing Example 3.5.3 to Banach spaces we get the following computation: let $I$ and $J$ be two countable sets, then

$$\bigoplus_{I} \mathbb{Q}_p \otimes^L_{\mathbb{Q}_p, \square} \bigoplus_{J} \mathbb{Q}_p = \bigoplus_{I \times J} \mathbb{Q}_p.$$  \hfill (3.10)
To prove this, since $\bigoplus_i Q_p = \left(\bigoplus_i Z_p\right)_{\overline{1}}$ it suffices to do the analogue computation for $Z_p$. By Example 3.5.2, the ring $Z_p$ is an idempotent solid $Z$-algebra, and so the $Z$-solid or $Z_p$-solid tensor products are the same. Then, Example 3.5.3 implies that the solid tensor product

$$\bigoplus_i Z_p \otimes^L \bigoplus_j Z_p$$

is $p$-adically complete, and so it is equal to

$$R\lim_{\leftarrow n}(\bigoplus_i Z/p^n \otimes^L \bigoplus_j Z/p^n) = R\lim_{\leftarrow n} \bigoplus_{i \times j} Z/p^n = \bigoplus_{i \times j} Z_p.$$  

For a more direct proof of this fact see [RJRC22, Lemma 3.13].

Example 3.5.5 (Tensor product Fréchet spaces). A Fréchet $Q_p$-vector space is by definition a sequential limit $F = \varprojlim V_n$ of Banach spaces, in particular they are naturally solid $Q_p$-vector spaces. If $G = \varprojlim W_n$ is another Fréchet space then

$$F \otimes^L G = \varprojlim_n (V_n \otimes^L W_n)$$

is the projective tensor product in classical functional analysis. In particular, we have that for $I$ and $J$ countable sets we get

$$\prod_i Q_p \otimes^L \prod_j Q_p = \prod_{i \times j} Q_p.$$  

For a proof of this fact see for example [RJRC22, Lemma 3.28].

4. Analytic rings

The building blocks of algebraic geometry are given by commutative rings. In analytic geometry the building blocks are the so called "analytic rings". The notion of analytic ring arises from the following desiderata:

- An analytic ring $A$ should have an underlying "topological" or condensed ring $A^\bullet$.
- An analytic rings $A$ should be endowed with a category of complete $A$-modules $\text{Mod}_A$, and with a complete tensor product $\otimes_A$.

In the next section we introduce analytic rings and prove some of their most fundamental properties. We will see how the new light foundations of the theory help to construct new examples of analytic rings.

4.1. First definitions and properties. We want to define building blocks for analytic geometry for which we can naturally attach a category of "complete modules". It turns out that in condensed mathematics a category of complete modules for a condensed ring is additional datum; given a condensed ring $A$ there could be very different ways to complete condensed $A$-modules, and none of them should have a preference. Nonetheless, once a category of "complete modules" is fixed, being a complete module should be just a property.

On the other hand, derived algebraic geometry [Lur04, Toe14] has shown that the correct framework to study geometric properties of algebraic varieties such as intersections is within higher category theory. In analytic geometry the requirement of higher category theory and higher algebra (taken in the form of [Lur09, Lur17, Lur18]) is even more notorious: even open localizations of rigid or complex spaces are not going to be flat. In particular, the only way to obtain actually useful new descent results is by looking at the $\infty$-derived categories of modules.

This desiderata for the notion of analytic ring is formalized in the following definition (see [CS20, Definition 12.1 and Proposition 12.20] and [Man22, Definition 2.3.1]).
Definition 4.1.1 (Analytic ring). An uncompleted analytic ring is a pair $A = (A^p, \mathscr{D}(A))$ consisting on a condensed animated ring $A^p$ and a full subcategory $\mathscr{D}(A) \subset \mathscr{D}(A^p)$ of the $\infty$-category of condensed $A^p$-modules satisfying the following properties.

1. $\mathscr{D}(A)$ is stable under limits and colimits in $\mathscr{D}(A^p)$ and there is a left adjoint $F : \mathscr{D}(A^p) \to \mathscr{D}(A)$ for the inclusion.
2. $\mathscr{D}(A)$ is linear over $\mathscr{D}(\text{CondAb})[3]$ More precisely for all $C \in \mathscr{D}(\text{CondAb})$ and $M \in \mathscr{D}(A)$ the object $\mathsf{RHom}_\mathbb{Z}(C, M)$ is in $\mathscr{D}(A)$.
3. The left adjoint $F$ sends connective objects to connective objects. In particular, $\mathscr{D}(A)$ has a natural $t$-structure induced from $\mathscr{D}(A^p)$ (see Proposition 4.1.7).

- We say that $A$ is an analytic ring structure of $A^p$. Finally, we say that $A$ is an analytic ring if in addition $A^p \in \mathscr{D}(A)$. We often write $A \otimes_{A^p} -$ for the left adjoint $F$ (note the drop of derived notation).
- A morphism of analytic rings $f : A \to B$ is a morphism of animated condensed rings $f : A^p \to B^p$ such that the forgetful functor $f_* : \mathscr{D}(B^p) \to \mathscr{D}(A^p)$ sends $\mathscr{D}(B)$ to $\mathscr{D}(A)$.
- We let $\text{AnRing}^{\text{un}}$ denote the $\infty$-category of uncompleted analytic rings. Let $\text{AnRing} \subset \text{AnRing}^{\text{un}}$ be the full subcategory of (completed) analytic rings.

Remark 4.1.2. Condition (2) of Definition 4.1.1 is equivalent to the following:

(2') For all $C \in \mathscr{D}(A^p)$ and $M \in \mathscr{D}(M)$ then $\mathsf{RHom}_{A^p}(C, M)$ is in $\mathscr{D}(A)$.

Indeed, it suffices to check the condition (2’) and (2) on generators of $\mathscr{D}(A^p)$ and $\mathscr{D}(\text{CondAb})$ respectively. Then we can suppose without loss of generality that $C = A^p[S]$ or $C = \mathbb{Z}[S]$ for $S \in \text{Prof}^{\text{light}}$. In this case we have

$$\mathsf{RHom}_{A^p}(A^p[S], M) = \mathsf{RHom}_\mathbb{Z}(\mathbb{Z}[S], M).$$

Remark 4.1.3. Recall that in the new foundations we work with light profinite sets, and so for a condensed animated ring $A^p$ the category $\mathscr{D}(A^p)$ is presentable. In particular, condition (1) of Definition 4.1.1 implies that the category $\mathscr{D}(A)$ is an accessible localization of $\mathscr{D}(A^p)$, and so presentable by [Lur09, Proposition 5.5.4.15] (the small class of morphisms we invert can be taken as the maps $A^p[S] \to A[S]$ for $S \in \text{Prof}^{\text{light}}$).

Example 4.1.4. So far we have seen essentially only two examples of analytic rings.

1. The initial analytic ring is $Z = (Z, \mathscr{D}(\text{CondAb}))$, the ring of condensed integers. More generally, given $B$ a condensed animated ring, we let $B = (B, \mathscr{D}(B))$ denote the trivial analytic ring structure on $B$.
2. A more "complete" analytic ring is $Z_{\text{CI}} = (Z, \mathscr{D}(\text{Solid}))$, the ring of solid integers. Later in §5 we shall introduce more examples of analytic rings arising in solid geometry.
3. Other analytic rings are the liquid rings of [CS20] and the gaseous ring of Example 1.4, these rings are global in the sense that they define analytic ring structures over the subring $\mathbb{Z}[q] \subset \mathbb{Z}[[q]]$ of null-sequences that specializes to analytic ring structures over all type of local fields (reals, $p$-adics, and modulo $p$).
4. In Section 4.4 we discuss a general way to construct analytic rings. This addresses a problem in the previous foundations of condensed mathematics, namely, the difficulty of constructing analytic rings.

Condensed rings embed fully faithful into analytic rings via the trivial analytic ring structure.

---

[1] This condition implies that $\mathscr{D}(A)$ is actually enriched in condensed abelian groups. It can be heuristically thought as a suitable "continuity" or "condensed" condition for $\mathscr{D}(A)$.
Proposition 4.1.5. The functor $F : \text{Cond Ani Ring} \to \text{AnRing}^\text{un}$ mapping an animated condensed ring $A^\circ$ to $(A, \mathcal{D}(A^\circ))$ is fully faithful. Moreover, $F$ has a right adjoint mapping an uncompleted analytic ring $B$ to its underlying condensed ring $B^\circ$.

Proof. By definition, given two uncompleted analytic rings $A$ and $B$ the mapping space $\text{Map}_{\text{AnRing}^\text{un}}(A, B)$ is the full subspace of $\text{Map}_{\text{CondRing}}(A^\circ, B^\circ)$ such that the forgetful functor $\mathcal{D}(B^\circ) \to \mathcal{D}(A^\circ)$ sends complete objects to complete objects. If $A$ has the trivial analytic ring structure this condition is tautological, proving that

$$\text{Map}_{\text{AnRing}^\text{un}}(A^\circ, B) = \text{Map}_{\text{CondRing}}(A^\circ, B^\circ)$$

proving the fully-faithfulness and the adjunction. □

The category of complete modules of an uncompleted analytic ring has a natural symmetric monoidal structure.

Proposition 4.1.6 ([CS20 Proposition 12.4] and [Man22 Proposition 2.3.2]). The category $\mathcal{D}(A)$ has a unique symmetric monoidal structure $\otimes_A$ making $A \otimes_A - : \mathcal{D}(A^\circ) \to \mathcal{D}(A)$ symmetric monoidal. Moreover, given $A \to B$ a morphism of analytic rings, the functor

$$\mathcal{D}(A^\circ) \xrightarrow{B^\circ \otimes_A} \mathcal{D}(B^\circ) \xrightarrow{B \otimes_A} \mathcal{D}(B)$$

factors (uniquely) through a functor

$$\mathcal{D}(A^\circ) \xrightarrow{A \otimes_A} \mathcal{D}(A) \xrightarrow{B \otimes_A} \mathcal{D}(B).$$

The functor $B \otimes_A -$ is the left adjoint of the forgetful functor $\mathcal{D}(B) \to \mathcal{D}(A)$.

Proof. To show that $\mathcal{D}(A)$ has a natural symmetric monoidal structure such that $A \otimes_A -$ is symmetric monoidal, it suffices to show that the kernel $K$ of the completion functor is a $\otimes$-ideal by [NST18 Theorem 1.3.6]. Let $M \in \mathcal{D}(A^\circ)$ be such that $A \otimes_A M = 0$ and let $C \in \mathcal{D}(A^\circ)$. Then, for $N \in \mathcal{D}(A)$, we have

$$\text{RHom}_{A^\circ}(A \otimes_A (C \otimes_A M), N) = \text{RHom}_{A^\circ}(C \otimes_A M, N)$$

$$= \text{RHom}_{A^\circ}(M, \text{RHom}_{A^\circ}(C, N))$$

$$= \text{RHom}_{A^\circ}(A \otimes_A M, \text{RHom}_{A^\circ}(C, N))$$

$$= 0,$$

where the first two equalities are the obvious adjunctions, and the third equality follows since $\text{RHom}_{A^\circ}(C, N)$ is $A$-complete by (2) of Definition 4.1.1 (cf. Remark 4.1.2). The previous shows that $A \otimes_A (C \otimes_A M) = 0$ as wanted.

Now, in order to see that the composite

$$\mathcal{D}(A^\circ) \xrightarrow{B^\circ \otimes_A} \mathcal{D}(B^\circ) \xrightarrow{B \otimes_A} \mathcal{D}(B)$$

factors through $\mathcal{D}(A)$, it suffices to see that it kills the kernel of $A \otimes_A -$ (then it would be immediate that the resulting functor is symmetric monoidal). Let $M \in \mathcal{D}(A)$ be an object killed by $A$-completion and let $K \in \mathcal{D}(B)$, then

$$\text{RHom}_{B^\circ}(B \otimes_{B^\circ} (B^\circ \otimes_A M), K) = \text{RHom}_{B^\circ}(B^\circ \otimes_A M, K)$$

$$= \text{RHom}_{A^\circ}(M, K)$$

$$= \text{RHom}_{A^\circ}(A \otimes_A M, K)$$

$$= 0,$$

where the first three equalities are adjunctions, and the last follows since $K$ is an $A$-complete module by definition of analytic ring. □

Completion of modules for analytic rings can be detected at the level of cohomology groups.
Proposition 4.1.7 ([CS20 Proposition 12.4]). Let $A$ be an analytic ring. An object $M \in \mathcal{D}(A^\op)$ is $A$-complete if and only if $\pi_i(M) = H^{-i}(M)$ is $A$-complete for all $i \in \mathbb{Z}$.

Proof. Let us first show the statement for connective objects (i.e. concentrated in positive homological degrees). Let $M \in \mathcal{D}(A)_{\geq 0}$ and consider the fiber sequence

$$\pi_{\geq 1} M \to M \to \pi_0 M.$$ 

Taking completions we get a fiber sequence

$$A \otimes A^\op (\pi_{\geq 1} M) \to M \to A \otimes A^\op (\pi_0 M).$$

Since completion preserves connective objects, taking $\geq 1$-truncations we get a map

$$A \otimes A^\op (\pi_{\geq 1} M) \to \pi_{\geq 1} M$$

which exhibits $\pi_{\geq 1} M$ as a retract of $A \otimes A^\op (\pi_{\geq 1} M)$. Since $\mathcal{D}(A)$ is stable under colimits we deduce that $\pi_{\geq 1} M$ and so $\pi_0 (M)$ are in $\mathcal{D}(A)$. An inductive argument shows that $\pi_i (M)$ is $A$-complete for all $i \geq 0$. Conversely, let $M \in \mathcal{D}_{\geq 0}(A^\op)$ be such that all its homotopy groups $\pi_i M$ are $A$-complete. Then $M = \varinjlim \tau_{\leq n} M$ is the limit of its Postnikov tower. By induction, each truncation $\tau_{\leq n} M$ is $A$-complete and then so is $M$ since $\mathcal{D}(A)$ is stable under limits.

We now prove the general case. Let $M \in \mathcal{D}(A)$, then we can write

$$M = \varinjlim_n \tau_{\geq -n} M,$$

and by the connective case it suffices to show that each $\tau_{\geq -n} M$ is $A$-complete. Since $A$-completion preserves connective objects, $\tau_{\geq -n} M$ is a retract of $A \otimes A^\op (\tau_{\geq -n} M)$, and so $A$-complete since $\mathcal{D}(A)$ is stable under colimits. Conversely, suppose that $M \in \mathcal{D}(A^\op)$ is such that $\pi_i (M)$ is $A$-complete for all $i \in \mathbb{Z}$. By the connective case we know that $\tau_{\geq -n} M$ is $A$-complete for all $n \in \mathbb{N}$. The proposition follows by writing $M = \varinjlim_n \tau_{\geq -n} M$. \qed

Our next goal is to show that analytic rings admit small colimits. As a first approximation let us show that uncompleted analytic rings have small colimits. First we will recall induced analytic structures [Man22, Definition 2.3.13].

Lemma 4.1.8 (Induced analytic structure). Let $A$ be an uncomplete analytic ring and let $B$ be an animated $A^\op$-algebra. Then there is a natural induced analytic structure $B_{A/}$ on $B$ such that $\mathcal{D}(B_{A/}) \subset \mathcal{D}(B)$ is the full subcategory of $B$-modules whose underlying $A^\op$-module is $A$-complete. The uncompleted analytic ring $B_{A/}$ is the pushout $A \otimes A^\op B$, where $A^\op$ and $B$ are endowed with the trivial analytic ring structure.

Proof. We want to see that $B_{A/}$ defines an (uncompleted) analytic ring structure on $B$. Stability under limits and colimits is clear since the forgetful functor $\mathcal{D}(B) \to \mathcal{D}(A^\op)$ commutes with limits and colimits. On the other hand, the inclusion $\mathcal{D}(B_{A/}) \to \mathcal{D}(B)$ has by left adjoint

$$B_{A/} \otimes_B - = A \otimes A^\op -$$

which still sends $B$-modules to $B$-modules as $A \otimes A^\op -$ is symmetric monoidal. Indeed, let $C \in \mathcal{D}(B^\op)$ and $K \in \mathcal{D}(B)$. We have a natural equivalence of $B^\op$-modules thanks to the Barr construction

$$C = B^\op \otimes_{B^\op} C = \varinjlim_{[n] \in \Delta^\op} B^\op \otimes_{A^\op} n + 1 \otimes_{A^\op} C.$$
Therefore,
\[
R\text{Hom}_B(C, K) = R\text{Hom}_B(\lim_{[n] \in \Delta} B^{\otimes A^\circ n+1} \otimes A^\circ C, K)
\]
\[
= \lim_{[n] \in \Delta} R\text{Hom}_B(B^{\otimes A^\circ n+1} \otimes A^\circ C, K)
\]
\[
= \lim_{[n] \in \Delta} R\text{Hom}_{A^\circ}(B^{\otimes A^\circ n} \otimes A^\circ C, K)
\]
\[
= \lim_{[n] \in \Delta} R\text{Hom}_{A^\circ}(A^{\otimes A^\circ B} \otimes_A (A \otimes A^\circ C), K)
\]

where in the first equivalence we use the Barr construction of the tensor product, the second equivalence follows since \(R\text{Hom}\) commutes with limits, the third follows by \(\otimes\)-adjunction, the fourth follows from adjunction of \(A\)-completion and the fact that \(K\) is \(A\)-complete. On the other hand, the same computation shows that
\[
R\text{Hom}_B(A \otimes A^\circ C, K) = \lim_{[n] \in \Delta} R\text{Hom}_{A^\circ}((A \otimes A^\circ B)^{\otimes A^\circ n} \otimes_A (A \otimes A^\circ C), K)
\]
\[
= \lim_{[n] \in \Delta} R\text{Hom}_{A^\circ}(A \otimes A^\circ (B^{\otimes A^\circ n} \otimes A^\circ C), K)
\]
\[
= R\text{Hom}_B(C, K),
\]
where the second equality follows since \(A\)-completion is symmetric monoidal and idempotent. This proves that \(B_{A^\circ} \otimes_B C = A \otimes A^\circ C\) as wanted.

Stability under \(R\text{Hom}(C, -)\) for \(C \in \mathcal{D}(\text{CondAb})\) is obvious. It is also clear that the left adjoint \(B_{A^\circ} \otimes_B -\) sends connective objects to connective objects. Thus we have proven that \(B_{A^\circ}\) is an analytic ring.

Let us now check that \(B_{A^\circ} = A \otimes A^\circ B\) as uncompleted analytic rings. Let \(C\) be an uncomplete analytic ring. Since \(B\) and \(A^\circ\) have the trivial analytic ring structure, Proposition 4.1.5 implies that a map \(B \to C\) is just given by a map of condensed rings \(B \to C^\circ\). Thus, it suffices to see that the following diagram of mapping spaces is cartesian
\[
\begin{array}{ccc}
\text{Map}_{\text{AnRing}^\text{uc}}(B_{A^\circ}, C) & \to & \text{Map}_{\text{CondRing}}(B, C^\circ) \\
\downarrow & & \downarrow \\
\text{Map}_{\text{AnRing}^\text{uc}}(A, C) & \to & \text{Map}_{\text{CondRing}}(A^\circ, C^\circ).
\end{array}
\] (4.1)

The bottom horizontal map of (4.1) is an inclusion. Then the pullback \(\mathcal{C}\) of (4.1) is the full subspace of \(\text{Map}_{\text{CondRing}}(B, C^\circ)\) consisting on those maps \(B \to C^\circ\) of \(A^\circ\)-algebras such that the forgetful functor \(\mathcal{D}(C^\circ) \to \mathcal{D}(B^\circ)\) sends \(C\)-complete objects to \(A\)-complete modules. But this is by definition the mapping space \(\text{Map}_{\text{AnRing}^\text{uc}}(B_{A^\circ}, C)\), proving what we wanted. \[\square\]

A second important kind of colimit of uncompleted analytic rings is obtained by taking intersections of analytic ring structures.
Lemma 4.1.9. Let $A^\circ$ be a condensed animated ring and let $\{A_i\}_{i \in I}$ be a diagram of (uncompleted) analytic ring structures over $A^\circ$. Then the pair $B = (A^\circ, \bigcap_i \mathcal{D}(A_i))$ is an (uncompleted) analytic ring representing the colimit $\lim_{\rightarrow i} A_i$ in the category $\text{AnRing}^{(un)}_{A^\circ/}$ of (uncompleted) analytic rings over $A^\circ$.

Proof. Let $B$ denote the pair $(A^\circ, \bigcap_i \mathcal{D}(A_i))$ where the intersection takes place in $\mathcal{D}(A^\circ)$. Note that conditions (1)-(3) of Definition 4.1.1 are stable under intersection; conditions (2) and (3) are obvious once (1) is proven. Stability under limits and colimits in (1) is clear. The existence of the left adjoint in (1) follows from the adjoint functor theorem [Lur09, Corollary 5.5.2.9]. Indeed, since all the functors involved in the diagram $I$ are accessible localizations of $\mathcal{D}(A^\circ)$, all the categories $\mathcal{D}(A_i)$ are presentable by Remark 4.1.3, and then so is its intersection by [Lur09, Theorem 5.5.3.18]. Moreover, if $A^\circ$ is $A_i$-complete for all $i$, it is also $B$-complete proving that $B$ is an analytic ring if all the $A_i$ are so.

It is left to show that $B$ is the colimit of the diagram $A_i$ in the category of (uncompleted) analytic rings over $A^\circ$. This follows from the fact that for any $C \in \text{AnRing}^{(un)}$ the maps

$$\text{Map}_{\text{AnRing}^{(un)}}(A_i, C) \to \text{Map}_{\text{CondRing}}(A_i^\circ, C^\circ)$$

are fully-faithful embeddings for all $i$, and then so its limit. Then, the limit $\lim_{\leftarrow i} \text{Map}_{\text{AnRing}^{(un)}}(A_i, C)$ over $\text{Map}_{\text{CondRing}}(A_i^\circ, C^\circ)$ is the full-subanima of $\text{Map}_{\text{CondRing}}(A^\circ, C^\circ)$ whose connected components are those maps $A^\circ \to C^\circ$ such that the forgetful functor sends $C$-complete modules to $A_i$-complete modules for all $i$. This is exactly the mapping space $\text{Map}_{\text{AnRing}^{(un)}}(B, C)$ proving what we wanted. \(\square\)

We can finally prove the existence of colimits in uncomplete analytic rings.

Proposition 4.1.10. The category $\text{AnRing}^{(un)}$ of uncompleted analytic rings have small colimits. More precisely, let $\{A_i\}_{i \in I}$ be a diagram of uncompleted analytic rings. Then $B = \lim_{\rightarrow i} A_i$ is the uncompleted analytic ring with underlying ring $B^\circ = \lim_{\rightarrow i} A_i^\circ$ and with category of complete modules $\mathcal{D}(B) \subset \mathcal{D}(A^\circ)$ given by those $B^\circ$-modules $M$ whose restrictions to an $A_i^\circ$-module is $A_i$-complete for all $i$.

Proof. First, let us show that the pair $B = (B^\circ, \mathcal{D}(B))$ constructed in the statement of the proposition is an analytic ring. This follows from the fact that $B$ can be written as the colimit

$$B = \lim_{\rightarrow i} B^\circ_{A_i/},$$

of uncompleted analytic ring structures over $B^\circ = \lim_{\rightarrow i} A_i^\circ$ (Proposition 4.1.9), where $B^\circ_{A_i/}$ is the induced analytic ring structure of Lemma 4.1.8.

Let us now consider the underlying diagram of condensed animated rings $\{A_i^\circ\}_{i \in I}$. Let $C \in \text{AnRing}^{(un)}$. By definition of the category of analytic rings the limit

$$\lim_{\leftarrow i} \text{Map}_{\text{AnRing}^{(un)}}(A_i^\circ, C)$$

(4.2)

is a full-subanima of the space

$$\lim_{\leftarrow i} \text{Map}_{\text{AnRing}^{(un)}}(A_i^\circ, C^\circ) = \text{Map}(B^\circ, C^\circ).$$

Furthermore, it is the full subanima of connected components consisting on those maps $B^\circ \to C^\circ$ for which a complete $C$-module is $A_i$-complete, equivalently, for which a complete $C$-module is $B^\circ_{A_i/}$ complete. This shows that (4.2) is the full anima $\text{Map}_{\text{AnRing}^{(un)}}(B, C) \subset \text{Map}(B^\circ, C^\circ)$, proving that $B = \lim_{\rightarrow i} A_i$ as wanted. \(\square\)

A first consequence of the previous lemma is the stability of analytic rings under sifted colimits in the category of uncompleted analytic rings.
Corollary 4.1.11. The \( \infty \)-category \( \text{AnRing} \) of analytic rings is stable under sifted colimits in \( \text{AnRing}^{\text{un}} \). Moreover, let \( B = \lim_{\rightarrow i} A_i \) be a sifted colimit of uncompleted analytic rings. Then for \( S \in \text{Prof}^{\text{light}} \) we have
\[
B[S] = \lim_{\rightarrow i} A_i[S]
\]

Proof. It suffices to show the second claim, namely, if the terms in the sifted colimits are analytic rings we have
\[
B[*] = \lim_{\rightarrow i} A_i[*] = \lim_{\rightarrow i} A_i^p = B^p,
\]
proving that \( B^p \) is \( B \)-complete. Let \( S \in \text{Prof}^{\text{light}} \) and consider the \( B^p \)-module \( M[S] = \lim_{\rightarrow i} A_i[S] \). It suffices to show that \( M[S] \) is \( B \)-complete, namely, for \( C \in \mathcal{D}(B) \) we have
\[
R\text{Hom}_{B^p}(M[S], C) = \lim_{\rightarrow i} R\text{Hom}_{A_i}(A_i[S], C) = R\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], C).
\]

We have to show that \( M[S] \) is \( B_{A_i/} \)-complete for all \( i \). Let us first argue when \( I \) is filtered. Fix \( j \in I \), for any \( i \geq j \) the module \( A_i[S] \) is \( A_j \)-complete and taking colimits on \( i \) one gets that \( M[S] \) is \( B_{A_j/} \)-complete. Since the previous hold for all \( j \) one deduces that \( M[S] \) is \( B \)-complete. Let us now consider a general sifted diagram \( \{A_i\}_{i \in I} \). We have then a sifted diagram \( \{B_{A_i/}\}_{i \in I} \) of analytic ring structures of \( B^p \). Note that the mapping space between two analytic ring structures \( B' \) and \( B'' \) over \( B^p \) is either contractible or empty, depending whether \( \mathcal{D}(B'') \subset \mathcal{D}(B') \) or not. Therefore, there is a surjective map of categories \( \pi : I \to I' \) with \( I' \) filtered, such that \( \{B_{A_i/}\}_{i \in I} \) can be refined to \( \{B_{A_{i'/}}\}_{i' \in I'} \). In particular, for \( C \in \mathcal{D}(B^p) \) we have
\[
C \otimes_{B^p} B = \lim_{\rightarrow i} C \otimes_{B^p} B_{A_i/}.
\]

Finally, we get that
\[
M[S] \otimes_{B^p} B = \lim_{\rightarrow i} (A_i[S] \otimes_{A_i^p} B)
\]
\[
= \lim_{\rightarrow i} (A_i[S] \otimes_{A_i^p} \lim_{\rightarrow j} B_{A_j/})
\]
\[
= \lim_{\rightarrow i} (A_i[S] \otimes_{A_i^p} B_{A_i/})
\]
\[
= \lim_{\rightarrow i} (A_i \otimes_{A_i^p} (A_i[S] \otimes_{A_i^p} \lim_{\rightarrow j} B_{A_j/}))
\]
\[
= \lim_{\rightarrow i} (A_i \otimes_{A_i^p} (A_i[S] \otimes_{A_i^p} \lim_{\rightarrow j} A_{j}))
\]
\[
= \lim_{\rightarrow i} (A_i \otimes_{A_i^p} (A_i[S] \otimes_{A_i^p} A_{i}))
\]
\[
= \lim_{\rightarrow i} A_i[A[S]
\]
\[
= M[S].
\]

where in the second equality we use [4.3], in the first, third and sixth equalities we use that \( I \) is sifted (so the diagonal \( I \to I \times I \) is cofinal), and the rest follows from the definitions. \( \square \)

In order to show that analytic rings admit arbitrary colimits we first need to discuss completions of analytic rings.

**Theorem 4.1.12 ([Man22 Proposition 2.3.12]).** The functor \( \text{AnRing} \to \text{AnRing}^{\text{un}} \) has a left adjoint \( A \mapsto A^\circ \) called the "completion functor". We have \( \mathcal{D}(A) = \mathcal{D}(A)^\circ \) and \( A^\circ \circ = A \otimes_{A^\circ} A^\circ \) is the \( A \)-completion of \( A^\circ \) (i.e. the unit in \( \mathcal{D}(A) \)). In particular, \( \text{AnRing} \) admits small colimits. A
diagram \{ A_i \}_{i} \text{ of analytic rings has colimit } B^= \text{ where } B = \lim_{\longrightarrow} A_i \text{ is the colimit in the category of uncompleted analytic rings.}

**Sketch of the proof.** We will prove a weaker version of the theorem where "animated ring" gets replaced by "commutative or \(E_\infty\)-ring". Indeed, the difficult part of the theorem is to show that the unit \(A^=\) has a natural animated ring structure. This will be handled in the next section.

Let \(B\) be an analytic ring and \(A\) an uncomplete analytic ring. By definition, \(\text{Map}_{\text{AnRing^{un}}}(A, B)\) is the full subanima of maps \(\text{Map}_{\text{CAlg}(D(\text{Cond}))}(A^\circ, B^\circ)\) of commutative condensed algebras such that the forgetful functor

\[ D(B^\circ) \rightarrow D(A^\circ) \]

sends \(D(B)\) to \(D(A)\). By [Lur17, Corollary 4.8.5.21] the space \(\text{Map}_{\text{CAlg}(D(\text{Cond}))}(A^\circ, B^\circ)\) is naturally equivalent to the space of \(D(\text{CondAb})\)-linear symmetric monoidal functors \(D(A^\circ) \rightarrow D(B^\circ)\). Therefore, \(\text{Map}_{\text{AnRing^{un}}}(A, B)\) gets identified with the full subcategory of symmetric monoidal functors as above that factor through

\[ D(A^\circ) \xrightarrow{B^\circ \otimes A^\circ} D(B^\circ) \]

\[ \downarrow^{A \otimes A^\circ} \downarrow^{B \otimes B^\circ} \]

\[ D(A) \longrightarrow D(B). \]

Since both \(D(A)\) and \(D(B)\) are localizations of \(D(A^\circ)\) and \(D(B^\circ)\) respectively, the space \(\text{Map}_{\text{AnRing^{un}}}(A, B)\) is naturally equivalent to the space of \(D(\text{CondAb})\)-linear symmetric monoidal functors \(D(A) \rightarrow D(B)\), which is also clearly equivalent to \(\text{Map}_{\text{AnRing}}(A^=, B)\), proving the desired adjunction.

The last claim about the computation of the colimit of analytic rings follows directly from the existence of the left adjoint \((-)^=\).

\[ \square \]

**References**


