EE363 Prof. S. Boyd

Linear quadratic Lyapunov theory

Lyapunov equations

We assume $A \in \mathbf{R}^{n\times n}$, $P = P^T \in \mathbf{R}^{n\times n}$. It follows that $Q = Q^T \in \mathbf{R}^{n\times n}$.

Continuous-time linear system: for $\dot{x} = Ax$, $V(z) = z^T P z$, we have $\dot{V}(z) = -z^T Q z$, where P, Q satisfy (continuous-time) Lyapunov equation $A^T P + PA + Q = 0$.

Discrete-time linear system: for $x(t+1) = Ax(t)$, $V(z) = z^T P z$, we have $\Delta V(z) = -z^T Q z$, where P, Q satisfy (discrete-time) Lyapunov equation $A^T P A - P + Q = 0$.

Lyapunov theorems

- If $P > 0$, $Q > 0$, then system is (globally asymptotically) stable.
- If $P > 0$, $Q \geq 0$, and (Q, A) observable, then system is (globally asymptotically) stable.
- If $P > 0$, $Q \geq 0$, then all trajectories of the system are bounded
- If $Q \geq 0$, then the sublevel sets $\{z \mid z^T P z \leq a\}$ are invariant. (These are ellipsoids if $P > 0$.)
- If $P \not\geq 0$ and $Q \geq 0$, then A is not stable.

Converse theorems

- If A is stable, there exists a quadratic Lyapunov function $V(z) = z^T P z$ that proves it, *i.e.*, there exists $P > 0$, $Q > 0$ that satisfies the (continuous- or discrete-time) Lyapunov equation.
- If A is stable and $Q \geq 0$, then $P \geq 0$.
- If A is stable, $Q \geq 0$, and (Q, A) observable, then $P > 0$.

Lyapunov equation solvability conditions

- The discrete-time Lyapunov equation has a unique solution P, for any $Q = Q^T$, if and only if $\lambda_i(A)\lambda_j(A) \neq 1$, for $i, j = 1, \ldots, n$.
- If A is stable, Lyapunov equation has a unique solution P, for any $Q = Q^T$.

Integral (sum) solution of Lyapunov equation

- If $\dot{x} = Ax$ is (globally asymptotically) stable and $Q = Q^T$,

$$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$$

  is the unique solution of the Lyapunov equation $A^T P + PA + Q = 0$.

- If $x(t+1) = Ax(t)$ is (globally asymptotically) stable and $Q = Q^T$,

$$P = \sum_{t=0}^{\infty} (A^T)^t Q A^t$$

  is the unique solution of the Lyapunov equation $A^T PA - P + Q = 0$.