Supplementary Material for "Top Trading Cycles in Two-Sided Matching Markets: An Irrelevance of Priorities in Large Markets"

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Yeon-Koo Che* Olivier Tercieux†

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S.1 Missing arguments for the proof of Theorem 1

S.1.1 Computation of β

To compute β, we first establish the following lemma.

S.1.1.1 Useful computational lemma

For the next result, consider agents I' and objects O' such that $|I'| = |O'| = m > 0$. We say a mapping $f = h \circ g$ is a **bipartite bijection**, if $g : I' \to O'$ and $h : O' \to I'$ are both bijections. A **cycle** of a bipartite bijection is a cycle of the induced digraph. Note that a bipartite bijection consists of disjoint cycles. A **random bipartite bijection** is a (uniform) random selection of a bipartite bijection from the set of all bipartite bijections. The following result will prove useful for a later analysis.

Lemma S1. *Fix sets I' and O' with $|I'| = |O'| = m > 0$, and a subset $K \subset I' \cup O'$, containing $a \geq 0$ vertices in I' and $b \geq 0$ vertices in O'. The probability that each cycle in a random bipartite bijection contains at least one vertex from K is*

$$\frac{a+b}{m} - \frac{ab}{m^2}.$$

*Department of Economics, Columbia University, USA. Email: yeonkooche@gmail.com.

†Department of Economics, Paris School of Economics, France. Email: tercieux@pse.ens.fr.

Proof. We begin with a few definitions. A **permutation** of X is a bijection $f : X \to X$. A **cycle** of a permutation is a cycle of the digraph induced by the permutation. A permutation consists of disjoint cycles. A **random permutation** chooses uniform randomly a permutation f from the set of all possible permutations. Our proof will invoke following result:

Fact 1 (Lovasz (1979) Exercise 3.6). *The probability that each cycle of a random permutation of a finite set X contains at least one element of a set $Y \subset X$ is $|Y|/|X|$.*

To begin, observe first that a bipartite bijection $h \circ g$ induces a permutation of set I'. Thus, a random bipartite bijection defined over $I' \times O'$ induces a random permutation of I'. To compute the probability that each cycle of a random bipartite bijection $h \circ g$ contains at least one vertex in $K \subset I' \times O'$, we shall apply Fact 1 to this induced random permutation of I'.

Indeed, each cycle of a random bipartite bijection contains at least one vertex in $K \subset I' \times O'$ if and only if each cycle of the induced random permutation of I' contains either a vertex in $K \cap I'$ or a vertex in $I' \setminus K$ that points to a vertex in $K \cap O'$ in the original random bipartite bijection. Hence, the relevant set $Y \subset I'$ for the purpose of applying Fact 1 is a random set that contains $|K \cap I'| = a$ vertices of the former kind and Z vertices of the latter kind.

The number Z is random and takes a value z, $\max\{b-a, 0\} \le z \le \min\{m-a, b\}$, with probability:

$$\Pr\{Z = z\} = \frac{\binom{m-a}{z}\binom{a}{b-z}}{\binom{m}{b}}.$$

This formula is explained as follows. $\Pr\{Z = z\}$ is the ratio of the number of bipartite bijections having exactly z vertices in $I' \setminus K$ pointing toward $K \cap O'$ to the total number of bipartite bijections.

Note that since we consider bipartite bijections, the number of vertices in I' pointing to the vertices in $K \cap O'$ must be equal to b. Focusing first on the numerator, we have to compute the number of bipartite bijections having exactly z vertices in $I' \setminus K$ pointing toward $K \cap O'$ and the remaining $b - z$ vertices pointing to the remaining $K \cap O'$. There are $\binom{m-a}{z}\binom{a}{b-z}$ ways one can choose z vertices from $I' \setminus K$ and $b - z$ vertices from $K \cap I'$. Thus, the total number of bipartite bijections having exactly z vertices in $I' \setminus K$ that point to $K \cap O'$ is $\binom{m-a}{z}\binom{a}{b-z}\upsilon$, where υ is the total number of bipartite bijections in which the b vertices thus chosen point to the vertices in $K \cap O'$. This gives us the numerator. As for the denominator, the total number of bipartite bijections having b vertices in I' pointing to $K \cap O'$ is $\binom{m}{b}$ (the number of ways b vertices are chosen from I'), multiplied by υ (the number of bijections in which the b vertices thus chosen point to the vertices in $K \cap O'$). Hence, the denominator is $\binom{m}{b}\upsilon$. Thus, we get the above formula.

Recall our goal is to compute the probability that each cycle of the random permutation induced by the random bipartite bijection contains at least one vertex in the random set Y, with $|Y| = a + Z$, where $\Pr\{Z = z\} = \frac{\binom{m-a}{z}\binom{a}{b-z}}{\binom{m}{b}}$. Applying Fact 1, then the desired probability is

$$
\begin{aligned}
\mathbb{E}\left[\frac{|Y|}{|I'|}\right] &= \sum_{z=\max\{b-a,0\}}^{\min\{m-a,b\}} \Pr\{Z = z\}\frac{a+z}{m} \\
&= \frac{a}{m} + \sum_{z=\max\{b-a,0\}}^{\min\{m-a,b\}} \Pr\{Z = z\}\frac{z}{m} \\
&= \frac{a}{m} + \sum_{z=\max\{b-a,0\}}^{\min\{m-a,b\}} \frac{\binom{m-a}{z}\binom{a}{b-z}}{\binom{m}{b}}\left(\frac{z}{m}\right) \\
&= \frac{a}{m} + \left(\frac{m-a}{m\binom{m}{b}}\right) \sum_{z=\max\{b-a,1\}}^{\min\{m-a,b\}} \binom{a}{b-z}\binom{m-a-1}{z-1} \\
&= \frac{a}{m} + \left(\frac{m-a}{m\binom{m}{b}}\right)\binom{m-1}{b-1} \\
&= \frac{a}{m} + \frac{b(m-a)}{m^2} \\
&= \frac{a+b}{m} - \frac{ab}{m^2},
\end{aligned}
$$

where the fifth equality follows from Vandermonde's identity. ∎

S.1.2 Completion of the computation of β

Recall that, given an arbitrary $F \in \mathcal{F}_{N_{i+1},k_{i+1}}$, $\beta(I_i, O_i, k_i^I, k_i^O; I_{i+1}, O_{i+1}, k_{i+1}^I, k_{i+1}^O)$ counts, the number of pairs (F', ϕ), $F' \in \mathcal{F}_{N_i,k_i}$, causing F to arise.

As mentioned in the main text, we can construct all such pairs by choosing a quadruplet (a, b, c, d) of four non-negative integers with $a + c = k_i^I$ and $b + d = k_i^O$,

(i) choosing c old roots from I_{i+1}, and similarly, d old roots from O_{i+1},

(ii) choosing a old roots from $I_i \backslash I_{i+1}$ and similarly, b old roots from $O_i \backslash O_{i+1}$,

(iii) choosing a partition into cycles of $N_i \backslash N_{i+1}$, each cycle of which contains at least one old root from (ii),[1]

[1]Within round i of TTC, one cannot have a cycle creating only with nodes that are not roots in the

(iv) choosing a mapping of the $k_{i+1}^I + k_{i+1}^O$ new roots to $N_i \backslash N_{i+1}$.

Hence, setting $n = |I_i|$, $o = |O_i|$ and $m = |I_i| - |I_{i+1}| = |O_i| - |O_{i+1}|$, the number of such pairs is computed as

$$\begin{aligned}
&\sum_{b+d=k^O} \sum_{a+c=k^I} \binom{n-m}{c}\binom{o-m}{d}\binom{m}{a}\binom{m}{b}\left(\frac{a+b}{m} - \frac{ab}{m^2}\right)(m!)^2 m^{\lambda^I+\lambda^O} \\
=&(m!)^2 m^{\lambda^I+\lambda^O} \times \Bigg(\sum_{b+d=k^O} \sum_{a+c=k^I} \binom{n-m}{c}\binom{o-m}{d}\binom{m-1}{a-1}\binom{m}{b} \\
&+ \sum_{b+d=k^O} \sum_{a+c=k^I} \binom{n-m}{c}\binom{o-m}{d}\binom{m}{a}\binom{m-1}{b-1} \\
&- \sum_{b+d=k^O} \sum_{a+c=k^I} \binom{n-m}{c}\binom{o-m}{d}\binom{m-1}{a-1}\binom{m-1}{b-1} \Bigg) \\
=&(m!)^2 m^{\lambda^I+\lambda^O} \times \left(\binom{o}{k^O}\binom{n-1}{k^I-1} + \binom{n}{k^I}\binom{o-1}{k^O-1} - \binom{n-1}{k^I-1}\binom{o-1}{k^O-1} \right) \\
=&\frac{(m!)^2 m^{\lambda^I+\lambda^O}}{no}\binom{n}{k^I}\binom{o}{k^O}(nk^O + ok^I - k^I k^O).
\end{aligned}$$

The first equality follows from Lemma S1, along with the fact that there are $(m!)^2$ possible bipartite bijections between $n-m$ agents and $o-m$ objects, and the fact that there are $m^{\lambda^I} m^{\lambda^O}$ ways in which new roots λ^I agents and λ^O objects could have pointed to $2m$ cyclic vertices (m on the individuals' side and m on the objects' side), the third equality follows from Vandermonde's identity, and the last equality follows from simplifying terms.

forest obtained at the beginning of round i. This is due to the simple fact that a forest is an acyclic graph. Thus, each cycle creating must contain at least one old root. Given that, by definition, these roots are eliminated from the set of available nodes in round $i+1$, these old roots that each cycle must contain must be from (ii).

S.1.3 Proof of Corollary 2

By symmetry, given $(n_1, o_1, k_1^I, k_1^O), ..., (n_i, o_i, k_i^I, k_i^O)$, the set (I_i, O_i) is chosen uniformly at random among all the $\binom{n}{n_i}\binom{o}{o_i}$ possible sets. Hence,

$$
\begin{aligned}
&\Pr((n_{i+1}, o_{i+1}, k_{i+1}^I, k_{i+1}^O)|(n_1, o_1, k_1^I, k_1^O), ..., (n_i, o_i, k_i^I, k_i^O))\\
=&\sum_{(I_i,O_i):|I_i|=n_i,|O_i|=o_i} \Pr\{(n_{i+1}, o_{i+1}, k_{i+1}^I, k_{i+1}^O)|(n_1, o_1, k_1^I, k_1^O), ..., (n_i, o_i, k_i^I, k_i^O), (I_i, O_i)\}\\
&\qquad\qquad \times \Pr\left\{(I_i, O_i) \mid (n_1, o_1, k_1^I, k_1^O), ..., (n_i, o_i, k_i^I, k_i^O)\right\}\\
=&\left(\sum_{(I_i,O_i):|I_i|=n_i,|O_i|=o_i} \Pr\{(n_{i+1}, o_{i+1}, k_{i+1}^I, k_{i+1}^O)|(n_1, o_1, k_1^I, k_1^O), ..., (I_i, O_i, k_i^I, k_i^O)\}\right)\frac{1}{\binom{n}{n_i}\binom{o}{o_i}}\\
=&\left(\sum_{\substack{(I_i,O_i):|I_i|=n_i,|O_i|=o_i\\(I_{i+1},O_{i+1}):|I_{i+1}|=n_{i+1},|O_{i+1}|=o_{i+1}}} \Pr\{(I_{i+1}, O_{i+1}, k_{i+1}^I, k_{i+1}^O)|(n_1, o_1, k_1^I, k_1^O), ..., (I_i, O_i, k_i^I, k_i^O)\}\right)\\
&\quad\times\frac{1}{\binom{n}{n_i}\binom{o}{o_i}}\\
=&\frac{1}{\binom{n}{n_i}\binom{o}{o_i}}\sum_{\substack{(I_i,O_i):|I_i|=n_i,|O_i|=o_i\\(I_{i+1},O_{i+1}):|I_{i+1}|=n_{i+1},|O_{i+1}|=o_{i+1}}} \Pr\{(I_{i+1}, O_{i+1}, k_{i+1}^I, k_{i+1}^O)|(I_i, O_i, k_i^I, k_i^O)\},
\end{aligned}
$$

where the second equality follows from the above reasoning and the last equality follows from the Markov property of $\{(I_i, O_i, k_i^I, k_i^O)\}$. The proof is complete by the fact that the last line, as shown in the proof of Lemma 3, depends only on $(n_{i+1}, o_{i+1}, k_{i+1}^I, k_{i+1}^O), (n_i, o_i, k_i^I, k_i^O))$.

S.1.4 Computation of transition probability

Before going through the algebra, we need the following lemma characterizing the number of spanning rooted forests.

Lemma S2 (Jin and Liu (2004))**.** *Let $V_1 \subset I$ and $V_2 \subset O$ where $|V_1| = \ell$ and $|V_2| = k$. The number of spanning rooted forests having k roots in V_1 and ℓ roots in V_2 is $f(n, o, k, \ell) := o^{n-k-1}n^{o-\ell-1}(\ell n + ko - k\ell)$.*

Now, we have

$$\begin{aligned}\Upsilon =& o^{k^I} n^{k^O} f(n, o, k^I, k^O)\\ =& o^{k^I} n^{k^O} \binom{n}{k^I}\binom{o}{k^O} o^{n-k^I-1} n^{o-k^O-1}(nk^O + ok^I - k^I k^O)\\ =& \binom{n}{k^I}\binom{o}{k^O} o^{n-1} n^{o-1}(nk^O + ok^I - k^I k^O).\end{aligned}$$

where the second equality follows from Lemma S2 and the last one uses Vandermonde's identity.

Θ is now computed as:

$$\begin{aligned}\Theta &= \binom{n}{m}\binom{o}{m} f(n-m, o-m, \lambda^I, \lambda^O)\beta(n, o, k^I, k^O; m, \lambda^I, \lambda^O)\\ &= \binom{n}{m}\binom{o}{m} f(n-m, o-m, \lambda^I, \lambda^O)\frac{(m!)^2 m^{\lambda^I+\lambda^O}}{no}\binom{n}{k^I}\binom{o}{k^O}(nk^O + ok^I - k^I k^O)\\ &= f(n-m, o-m, \lambda^I, \lambda^O)\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)\frac{m^{\lambda^I+\lambda^O}}{no}\binom{n}{k^I}\binom{o}{k^O}(nk^O + ok^I - k^I k^O).\end{aligned}$$

Collection the terms we obtain

$$\mathbf{P}(n, o, k^I, k^O; m, \lambda^I, \lambda^O) = \frac{\Theta}{\Upsilon} = \frac{1}{o^n n^o}\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right) m^{\lambda^I+\lambda^O} f(n-m, o-m, \lambda^I, \lambda^O).$$

Recall that the transition probability can be obtained by summing the expression over all possible (λ^I, λ^O)'s:

$$p_{n,o;m} := \sum_{0\le\lambda^I\le n-m, 0\le\lambda^O\le o-m} \mathbf{P}(n, o, k^I, k^O; m, \lambda^I, \lambda^O).$$

Hence, we obtain:

$$
\begin{aligned}
&\sum_{0\le\lambda^I\le n-m}\ \sum_{0\le\lambda^O\le o-m} m^{\lambda^I} m^{\lambda^O} f(n-m, o-m, \lambda^I, \lambda^O)\\
=&\sum_{0\le\lambda^I\le n-m}\ \sum_{0\le\lambda^O\le o-m} m^{\lambda^I} m^{\lambda^O} \binom{n-m}{\lambda^I}\binom{o-m}{\lambda^O}\times\\
&\qquad (o-m)^{n-m-\lambda^I-1}(n-m)^{o-m-\lambda^O-1}((n-m)\lambda^O+(o-m)\lambda^I-\lambda^I\lambda^O)\\
=&m\left(\sum_{0\le\lambda^I\le n-m}\binom{n-m}{\lambda^I}m^{\lambda^I}(o-m)^{n-m-\lambda^I}\right)\left(\sum_{1\le\lambda^O\le o-m}\binom{o-m-1}{\lambda^O-1}m^{\lambda^O-1}(n-m)^{o-m-\lambda^O}\right)\\
+&m\left(\sum_{1\le\lambda^I\le n-m}\binom{n-m-1}{\lambda^I-1}m^{\lambda^I-1}(o-m)^{n-m-\lambda^I}\right)\left(\sum_{0\le\lambda^O\le o-m}\binom{o-m}{\lambda^O}m^{\lambda^O}(n-m)^{o-m-\lambda^O}\right)\\
-&m^2\left(\sum_{1\le\lambda^I\le n-m}\binom{n-m-1}{\lambda^I-1}m^{\lambda^I-1}(o-m)^{n-m-\lambda^I}\right)\left(\sum_{1\le\lambda^O\le o-m}\binom{o-m-1}{\lambda^O-1}m^{\lambda^O-1}(n-m)^{o-m-\lambda^O}\right)\\
=&mo^{n-m}n^{o-m-1}+mo^{n-m-1}n^{o-m}-m^2o^{n-m-1}n^{o-m-1}\\
=&mo^{n-m-1}n^{o-m-1}(n+o-m),
\end{aligned}
$$

where the first equality follows from Lemma S2, and the third follows from the Binomial Theorem.

Multiplying the term $\frac{1}{o^n n^o}\left(\frac{n!}{(n-m)!}\right)\left(\frac{o!}{(o-m)!}\right)$, we get the formula stated in Theorem 1.

S.2 Number of agents matched at each stage of TTC

Consider an arbitrary mapping, $g : I \to O$ and $h : O \to I$, defined over our finite sets I and O. Note that such a mapping naturally induces a bipartite digraph with vertices $I \cup O$ and directed edges with the number of outgoing edges equal to the number of vertices, one for each vertex. In this digraph, $i \in I$ points to $g(i) \in O$ while $o \in O$ points to $h(o) \in I$. Such a mapping will be called a bipartite mapping. A **cycle** of a bipartite mapping is a cycle in the induced bipartite digraph, namely, distinct vertices $(i_1, o_1, ..., i_{k-1}, o_{k-1}, i_k)$ such that $g(i_j) = o_j, h(o_j) = i_{j+1}, j = 1, ..., k-1, i_k = i_1$. A **random bipartite mapping** selects a composite map $h \circ g$ uniformly from a set $\mathcal{H} \times \mathcal{G} = I^O \times O^I$ of all bipartite mappings. Note that a random bipartite mapping induces a random bipartite digraph consisting of vertices $I \cup O$ and directed edges emanating from vertices, one for each vertex. We say that a vertex in a digraph is **cyclic** if it is in a cycle of the digraph.

The following lemma states the number of cyclic vertices in a random bipartite digraph induced by a random bipartite mapping.

Lemma S3 (Jaworski (1985), Corollary 3)**.** *The number q of the cyclic vertices in a random bipartite digraph induced by a random bipartite mapping $g : I \to O$ and $h : O \to I$ has an expected value of*

$$\mathbb{E}[q] := 2 \sum_{i=1}^{\min\{o,n\}} \frac{(o)_i (n)_i}{o^i n^i},$$

and a variance of

$$8 \sum_{i=1}^{\min\{o,n\}} \frac{(o)_i (n)_i}{o^i n^i} i - \mathbb{E}[q] - \mathbb{E}^2[q],$$

where $(x)_j := x(x-1)\cdots(x-j-1)$.

It is clear that at the beginning of the first round of TTC, if there are n agents and o objects in the economy, the distribution of the number of individuals and objects assigned is the same as that of q. Appealing to Theorem 1 we can further obtain that *for any* round of TTC which begins with n agents and o objects remaining in the market, the number of individuals and objects assigned has the same distribution as q. Hence, the first and second moments of the number of individuals/objects matched at that round corresponds exactly to those in the above lemma. Jaworski (1985) also shows that asymptotically (as o and n grow) the expectation of q is $\sqrt{2\pi \frac{no}{n+o}}(1+o(1))$ while its variance is $(4-\pi)\frac{2no}{n+o}(1+o(1))$. Given the number n of individuals and o of objects available at the beginning of Stage t of TTC, if we denote X_t the number of agents and objects matched at that stage, we have that $\mathbb{E}[\frac{X_t}{\sqrt{2\pi \frac{no}{n+o}}}]$ converges to 1 as n grows while the variance of $\frac{X_t}{\sqrt{2\pi \frac{no}{n+o}}}$ converges to the constant $\frac{4-\pi}{\pi}$.

S.3 Number of Rounds Required for TTC and Shapley-Scarf TTC to Conclude

Frieze and Pittel (1995) analyze Shapley-Scarf TTC. They obtain a similar Markov chain result for Shapley-Scarf TTC. Our result allows us to compare the two Markov chains. Specifically, we can order the two chains in terms of likelihood ratio order. To see this, let us recall the transition probabilities of the Markov chain obtained by Frieze and Pittel (1995):

$$\hat{p}_{n;m} = \frac{n!}{n^m (n-m)!} \frac{m}{n}$$

By Theorem 1, we obtain (assuming $n = o$):

$$\begin{aligned} p_{n;m} \quad : \quad = p_{n,n;m} &= \left(\frac{m}{(n)^{2(m+1)}}\right)\left(\frac{n!}{(n-m)!}\right)^2 (2n-m) \\ &= \left(\frac{n!}{n^m(n-m)!}\right)^2 \left(\frac{m(2n-m)}{n^2}\right). \end{aligned}$$

Let us compare the two distributions in terms of likelihood ratio order. Fix $n \geq 1$ and any $m' \geq m$. It is easy to check that

$$\frac{\hat{p}_{n,m'}}{\hat{p}_{n,m}} = \frac{n^m(n-m)!}{n^{m'}(n-m')!}\frac{m'}{m}$$

while

$$\frac{p_{n,m'}}{p_{n,m}} = \left(\frac{n^m(n-m)!}{n^{m'}(n-m')!}\right)^2 \left(\frac{m'}{m}\right)\left(\frac{2n-m'}{2n-m}\right).$$

Now, observe that

$$\begin{aligned} \left(\frac{\hat{p}_{n,m'}}{\hat{p}_{n,m}}\right)^{-1} \frac{p_{n,m'}}{p_{n,m}} &= \left(\frac{1}{n^{m'-m}}\right)\left(\frac{(n-m)!}{(n-m')!}\right)\frac{(2n-m')}{(2n-m)} \\ &= \frac{(n-m)(n-m-1)...(n-m'+1)}{n^{m'-m}}\left(\frac{2n-m'}{2n-m}\right) \leq 1. \end{aligned}$$

This proves that for any n, the distribution $\hat{p}_{n,\cdot}$ dominates $p_{n,\cdot}$ in terms of likelihood ratio order. One can prove an interesting implication of this result: for each $t \geq 1$, the probability that TTC stops before Round t is smaller than the probability that Shapley-Scarf TTC stops before Round t. In other words, the random round at which TTC stops is (first order) stochastically dominated by that at which the Shapley-Scarf TTC stops.

S.4 The Number of Objects Assigned via Short Cycles

Recall the random sequence of forests, $F_1, F_2, ...$ is defined in the main text, where F_1 is a trivial unique forest consisting of $|I| + |O|$ trees with isolated vertices, forming their own roots. For any $i = 2, ...,$ we first create a random directed edge from each root of F_{i-1} to a vertex on the other side, and then delete the resulting cycles (these are the agents and objects assigned in round $i - 1$) and F_i is defined to be the resulting rooted forest.

We begin with the following question: If round k of TTC begins with a rooted forest F, what is the expected number of short-cycles that will form at the end of that round?

We will show that, irrespective of F, this expectation is bounded by 2. To show this, we will make a couple of observations.

To begin, let n_k be the cardinality of the set I_k of individuals in our forest F and let o_k be the cardinality of O_k, the set of F's objects. And, let $A \subset I_k$ be the set of roots on the individuals side of our given forest F and let $B \subset O_k$ be the set of its roots on the objects side. Let their cardinalities be a and b, respectively.

Now, observe that for any $(i,o) \in A \times B$, the probability that (i,o) forms a short-cycle is $\frac{1}{n_k}\frac{1}{o_k}$. For any $(i,o) \in (I_k \backslash A) \times B$, the probability that (i,o) forms a short-cycle is $\frac{1}{n_k}$ if i points to o and 0 otherwise. Similarly, for $(i,o) \in A \times (O_k \backslash B)$, the probability that (i,o) forms a short-cycle is $\frac{1}{o_k}$ if o points to i and 0 otherwise. Finally, for any $(i,o) \in (I_k \backslash A) \times (O_k \backslash B)$, the probability that (i,o) forms a short-cycle is 0 (by definition of a forest, i and o cannot be pointing to each other in the forest F). So, given the forest F, the expectation of the number S_k of short-cycles is

$$\begin{aligned}
\mathbb{E}\left[S_k | F_k = F\right] &= \mathbb{E}\left[\sum_{(i,o) \in I_k \times O_k} \mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} \middle| F_k = F\right] \\
&= \sum_{(i,o) \in I_k \times O_k} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&= \sum_{(i,o) \in A \times B} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&\quad + \sum_{(i,o) \in (I_k \backslash A) \times B} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&\quad + \sum_{(i,o) \in A \times (O_k \backslash B)} \mathbb{E}\left[\mathbf{1}_{\{(i,o) \text{ is a short-cycle}\}} | F_k = F\right] \\
&= \sum_{(i,o) \in A \times B} \Pr\{(i,o) \text{ is a short-cycle} | F_k = F\} \\
&\quad + \sum_{(i,o) \in (I_k \backslash A) \times B} \Pr\{(i,o) \text{ is a short-cycle} | F_k = F\} \\
&\quad + \sum_{(i,o) \in I_k \times (O_k \backslash B)} \Pr\{(i,o) \text{ is a short-cycle} | F_k = F\} \\
&\leq \frac{ab}{n_k o_k} + \frac{n_k - a}{n_k} + \frac{o_k - b}{o_k} \\
&= 2 - \frac{a o_k + b n_k - ab}{n_k o_k} \leq 2.
\end{aligned}$$

Observe that since $o_k \geq b$, the above term is smaller than 2. Thus, as claimed, we obtain the following result.[2]

Proposition S1. *If TTC round k begins with any forest F,*

$$\mathbb{E}\left[S_k \,|F_k = F\right] \leq 2.$$

Given that our upper bound holds for any forest F, we get the following corollary.

Corollary S1. *For any round k of TTC, $\mathbb{E}\left[S_k\right] \leq 2$.*

S.5 Proof of Proposition 1

We establish several lemmas before proving the proposition. Assume wlog that $|I| = n \leq o = |O|$ (the proof is symmetric when $n \geq o$). Let $\{n_t\}$ be the (random) sequence corresponding to the number of individuals at Step t of TTC. By our main result, this is a Markov chain. Let c_t be the number of cyclic vertices on the individual side obtained in the graph of TTC at Step t so that $n_{t+1} = n_t - c_t$ for each $t \geq 1$. In general, $n_t = n - \sum_{k=1}^{t-1} c_k$. Thus, $\mathbb{E}[n_t] = n - \sum_{k=1}^{t-1} \mathbb{E}[c_k]$. Finally, letting $\{o_t\}$ be the (random) sequence corresponding to the number of objects at Step t of TTC, we observe that $n_t \leq o_t$ for all $t \geq 1$ since $n \leq o$ (and the same number of individuals and objects are assigned at each step).

The following lemma shows that, if we start from any Step t_0 of TTC with a number of agents/objects $n_{t_0} \geq \delta n$, for any arbitrarily small $\delta > 0$, then with a significant probability, after a number of steps linear in $\sqrt{n}$ we will end up with fewer than δn agents remaining in the market.

Lemma S4. *Consider any Step $t_0 \geq 1$ of TTC. Fix any $\delta > 0$ and let $c := \frac{1}{\sqrt{\pi\delta}}$. There is $\gamma > 0$ such that $\liminf \Pr\{n_{t_0+c\sqrt{n}} \leq \delta n \,|n_{t_0} \geq \delta n\} > \gamma$ where γ does not depend on t_0.*

Proof. In the sequel, we condition on the event that $n_{t_0} \geq \delta n$. By the Markov chain property, we can view the process as starting with a number n_{t_0} of agents/objects. To avoid notational clutter, we suppress the dependence on the conditioning event $\{n_{t_0} \geq \delta n\}$ throughout. Assume that there is $\delta > 0$ such that $\limsup \Pr\{n_{t_0+c\sqrt{n}} > \delta n\} = 1$. Let us further assume that $\lim_{n\to\infty} \Pr\{n_{t_0+c\sqrt{n}} > \delta\} = 1$, taking a subsequence if necessary. Note that the event $\{n_{t_0+c\sqrt{n}} > \delta n\}$ implies that $n_t > \delta n$ for any $t_0 \leq t \leq t_0 + c\sqrt{n}$. Thus, for

[2]Note that the bound is pretty tight: if the forest F has one root on each side and each node which is not a root points to the (unique) root on the opposite side, the expected number of short-cycles given F is $\frac{1}{n_k o_k} + \frac{n_k - 1}{n_k} + \frac{o_k - 1}{o_k} \to 2$ as $n_k, o_k \to \infty$. Thus, the conditional expectation of s_k is bounded by 2 and, asymptotically, this bound is tight. However, we can show, using a more involved computation, that the unconditional expectation of s_k is bounded by 1. The details of the computation are available upon request.

each $t_0 \leq t \leq t_0 + c\sqrt{n}$, $\Pr\{n_t > \delta n\} \geq \Pr\{n_{t_0+c\sqrt{n}} > \delta n\} \to 1$ and so, since the lower bound on $\Pr\{n_t > \delta n\}$ does not depend on t, $\Pr\{n_t > \delta n\}$ goes to 1 uniformly across $t_0 \leq t \leq t_0 + c\sqrt{n}$. Now, by definition,

$$\mathbb{E}[c_t] = \mathbb{E}[c_t \,|n_t > \delta n] \Pr\{n_t > \delta n\} + \mathbb{E}[c_t \,|n_t \leq \delta n] \Pr\{n_t \leq \delta n\}$$

and so

$$\frac{\mathbb{E}[c_t]}{\mathbb{E}[c_t \,|n_t > \delta n]} = \Pr\{n_t > \delta n\} + \frac{\mathbb{E}[c_t \,|n_t \leq \delta n]}{\mathbb{E}[c_t \,|n_t > \delta n]} \Pr\{n_t \leq \delta n\}.$$

Thus, using the fact that $\Pr\{n_t > \delta n\}$ converges to 1 uniformly across any $t_0 \leq t \leq t_0 + c\sqrt{n}$, we obtain that $\frac{\mathbb{E}[c_t]}{\mathbb{E}[c_t|n_t > \delta n]}$ converges to 1 uniformly across $t_0 \leq t \leq t_0 + c\sqrt{n}$. So we must have that for any $\varepsilon > 0$, there is $N > 0$ and for any $n > N$,

$$\begin{aligned}
\mathbb{E}[c_t] \geq \left(1 - \frac{\varepsilon}{2}\right) \mathbb{E}[c_t \,|n_t > \delta n] &= \left(1 - \frac{\varepsilon}{2}\right) \mathbb{E}[c_t \,|n_t > \delta n, o_t > \delta n + (o - n)] \\
&\geq \left(1 - \frac{\varepsilon}{2}\right) \mathbb{E}[c_t \,|n_t = \delta n, o_t = \delta n] \\
&\geq (1 - \varepsilon) \sqrt{\pi \delta n} \\
&= (1 - \varepsilon) \sqrt{\pi \delta} \sqrt{n},
\end{aligned}$$

for any $t_0 \leq t \leq t_0 + c\sqrt{n}$, where the first equality comes from the simple observation that $o_t - n_t = o - n \geq 0$ for any period t and the last inequality comes from the Markov property together with the fact that $\lim \frac{\mathbb{E}[c_t|n_t = \delta n, o_t = \delta n]}{\sqrt{\pi \delta n}} \geq 1$ (Jaworski (1985), Theorem 9).

Importantly, note that the N exhibited above does not depend on the specific $t_0 \leq t \leq t_0 + c\sqrt{n}$.

Thus, for any $\varepsilon > 0$, there is N such that for any $n > N$, we have

$$\begin{aligned}
\mathbb{E}[n_{t_0+c\sqrt{n}}] &= \mathbb{E}\left[n_{t_0} - \sum_{k=t_0}^{t_0+c\sqrt{n}-1} c_k\right] \\
&\leq n - \sum_{k=t_0}^{t_0+c\sqrt{n}-1} \mathbb{E}[c_k] \\
&\leq n - \left(c\sqrt{n}\right)(1 - \varepsilon) \sqrt{\pi \delta} \sqrt{n} \\
&= n - (1 - \varepsilon)\, n = \varepsilon n.
\end{aligned}$$

In other words, $\lim_{n\to\infty} \mathbb{E}[n_{t_0+c\sqrt{n}}/n] = 0$. This in turn implies that $\lim_{n\to\infty} \Pr\{n_{t_0+c\sqrt{n}} \leq \delta n\} = 1$, a contradiction to our assumption that $\lim \Pr\{n_{t_0+c\sqrt{n}} > \delta n\} = 1$.

To recap, we obtain that there is $\gamma > 0$ such that $\lim \Pr\{n_{t_0+c\sqrt{n}} \leq \delta n \,|n_{t_0} \geq \delta n\} > \gamma$ (we now make explicit the conditioning event $\{n_{t_0} \geq \delta n\}$). That γ does not depend on the specific starting date t_0 comes from the Markov property of the random process $\{n_t\}$. ∎

Lemma S5. *Fix any $\delta > 0$ and let $c := \frac{1}{\sqrt{\pi\delta}}$. For any $\xi > 0$, for any $k \in \mathbb{N}$ large enough, $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$ for any n large enough.*

Proof. We know by the previous lemma that there is $\gamma > 0$ such that for n large enough, $\Pr\{n_{c\sqrt{n}} \leq \delta n\} > \gamma$. First, note that $\Pr\{n_{2c\sqrt{n}} \leq \delta n\} > \gamma + (1-\gamma)\gamma$. Indeed, because $\{n_t\}$ is a decreasing sequence, $\{n_{c\sqrt{n}} \leq \delta n\}$ implies $\{n_{2c\sqrt{n}} \leq \delta n\}$. Hence, we have

$$
\begin{aligned}
&\Pr\{n_{2c\sqrt{n}} \leq \delta n\} \\
&= \Pr\{n_{c\sqrt{n}} \leq \delta n\} \Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\, n_{c\sqrt{n}} \leq \delta n\} + \Pr\{n_{c\sqrt{n}} > \delta n\} \Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\, n_{c\sqrt{n}} > \delta n\} \\
&= \Pr\{n_{c\sqrt{n}} \leq \delta n\} + \Pr\{n_{c\sqrt{n}} > \delta n\} \Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\, n_{c\sqrt{n}} > \delta n\}
\end{aligned}
$$

Applying Lemma S4 for $t_0 = 1$, we know that, for n large enough, $\Pr\{n_{c\sqrt{n}} \leq \delta n\} > \gamma$. In addition, applying Lemma S4 for $t_0 = c\sqrt{n}$, we know that, for n large enough, $\Pr\{n_{2c\sqrt{n}} \leq \delta n \,\big|\, n_{c\sqrt{n}} > \delta n\} > \gamma$. Thus, we obtain that for any n large enough, $\Pr\{n_{2c\sqrt{n}} \leq \delta n\} \geq \gamma + (1-\gamma)\gamma$, as claimed.

Similar reasoning yields that for each $k \in \mathbb{N}$, there is N large enough so that

$$\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \sum_{\ell=1}^{k} (1-\gamma)^{\ell-1}\gamma = 1 - (1-\gamma)^k.$$

Note that the right-hand side is equal to the cumulative distribution at k of a geometric distribution with parameter γ. Clearly, this goes to 1 as k increases and so our argument is complete. Thus, if we fix any $\xi > 0$, we can find k large enough so that $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$ for any n large enough, as was to be proved. ∎

We are now ready to prove Proposition 1.

Proof. Fix any $\alpha > 0$ and $\xi < 1$, we claim that there is n large enough so that $\Pr\{\frac{T}{n} \leq \alpha\} > \xi$. Consider any $\delta \in (0, \alpha)$ and fix $k \in \mathbb{N}$ and $c = \frac{1}{\sqrt{\pi\delta}}$ in order to have $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$ for any n large enough; which is well-defined by Lemma S5. Note that $\{n_{kc\sqrt{n}} \leq \delta n\}$ implies that $T \leq kc\sqrt{n} + \delta n$. Because, $\delta < \alpha$, the term on the right-hand side of the inequality is smaller than αn when n is large enough. Thus, for n large enough, we obtain $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} \leq \Pr\{T \leq kc\sqrt{n} + \delta n\} \leq \Pr\{\frac{T}{n} \leq \alpha\}$. Now, because, for n large enough, $\Pr\{n_{kc\sqrt{n}} \leq \delta n\} > \xi$, we obtain that for n large enough, $\Pr\{\frac{T}{n} \leq \alpha\} > \xi$, as claimed. ∎

S.6 Proof of Theorem 2

S.6.1 Proof of Theorem 2 (a)

Recall our observation that any object assigned via long cycles is uniform-randomly assigned across individuals whom the object ranks below R_o^* (i.e., ranks larger than R_o^*). This in turn means that the rank enjoyed by each object $o \in \bar{O}$ is stochastically dominated by the uniform distribution across $\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$. To precisely characterize the implication of this observation, recall that R_o (resp. R_i) denote the rank enjoyed by object o (resp. enjoyed by individual i) under TTC. We also let an arbitrary vector $(x_k)_{k \in K}$ be denoted by $\mathbf{x}_K$. For instance, $\mathbf{R}_O$ stands for $\{R_o\}_{o \in O}$. We are now ready to present the corner stone for the proof of part (a) of Theorem 2.

Proposition S2. *Fix any $I' \subseteq I$ and $O' \subseteq O'' \subseteq O$. For any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,*

$$\Pr\left\{\mathbf{R}_{O'} \leq \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\} \geq \prod_{o \in O'} \Pr\left\{Y_o \leq \ell_o\right\} \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\}$$

where $\{Y_o\}_{o \in O'}$ is a collection of iid random variables where each Y_o follows the uniform distribution over $\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$. In addition, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,

$$\Pr\left\{\mathbf{R}_{O'} \leq \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\} \leq \prod_{o \in O'} \Pr\left\{U_o \leq \ell_o\right\} \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \bar{O} = O''\right\}$$

where $\{U_o\}_{o \in O'}$ is a collection of iid random variables where each U_o follows the uniform distribution over $\{1, ..., n\}$.

Roughly speaking, the proposition asserts that the distribution of the rank enjoyed by each object within $\bar{O}$ is "squeezed" (according to first-order stochastic dominance) in between uniform from $\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$ from above and uniform from $\{1, ..., n\}$ from below, independently of the distribution of the ranks enjoyed by the agents and the ranks enjoyed by the other objects in the set $\bar{O}$.

To prove Proposition S2, we start with the following lemma.

Lemma S6. *Fix any $O'' \subseteq O$. For any $O' \subseteq O''$, for any $\boldsymbol{\ell}_I := (\ell_i)_{i \in I}, \boldsymbol{\ell}_{O'} := (\ell_o)_{o \in O'}$ and $\mathbf{R}^*_{O'} := (r_o)_{o \in O'}$,*

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \bar{O} = O''\} = 0$$

if $\ell_o \leq r_o$ for some $o \in O'$ and is a strictly positive number which does not depend on $\boldsymbol{\ell}_{O'}$ otherwise.

Proof. In the sequel, to save on notation, we let $\mathcal{E}$ be $\{\bar{O} = O''\}$. We first note that

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} = 0.$$

if for some $o \in O'$, $\ell_o \leq r_o$. Indeed, by definition, o points to r_o when involved in a cycle. In addition, $o \in O' \subseteq \bar{O}$ implies that object o is assigned via a long cycle, hence, the individual o is matched to must have a priority rank strictly greater than r_o.

Now, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}'_{O'}$ satisfying $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}'_{O'} \gg \mathbf{r}_{O'}$, we argue that

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} = \Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}.$$

Indeed, fix a profile of preferences and priorities yielding $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$. For each object $o \in O'$, let i be the individual with rank ℓ'_o. Swap i and $k := \mathrm{TTC}(o)$ in o's priority ordering. Clearly, k has rank ℓ'_o at o. In addition, since for each object o, r_o (the individual o points to when involved in a cycle under the original profile) has a priority rank less than those of both i and k at the original profile (recall that by assumption $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}'_{O'} \gg \mathbf{r}_{O'}$), each step of the TTC algorithm remains unchanged after the swaps. Hence, $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is obtained. Thus, we have an injection from the set of profiles of preferences and priorities yielding $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ to the one yielding $\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$. Given the iid distribution of priority order, it follows that

$$\Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \leq \Pr\{\mathbf{R}_I = \boldsymbol{\ell}_I, \mathbf{R}_{O'} = \boldsymbol{\ell}'_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}.$$

A similar reasoning shows that the inequality holds in the other direction as well. ∎

Now, we can complete the proof of Proposition S2. Here again, in the sequel, to save on notation, we let $\mathcal{E}$ be $\{\bar{O} = O''\}$. By the above lemma, for any $O_1, O_2 \subseteq O'$ disjoint, whenever well-defined, $\Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is a positive number which does not depend on $\boldsymbol{\ell}_{O_2}$.[3] Hence, we can write

$$\begin{aligned} & \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\ = & \sum_{\boldsymbol{\ell}'_{O_2}} \Pr\left\{\mathbf{R}_{O_2} = \boldsymbol{\ell}'_{O_2} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}'_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\ = & \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \sum_{\boldsymbol{\ell}'_{O_2}} \Pr\left\{\mathbf{R}_{O_2} = \boldsymbol{\ell}'_{O_2} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\ = & \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}, \end{aligned} \tag{S1}$$

where $\boldsymbol{\ell}_{O_2}$ is an arbitrary profile under which the above conditional probability is well-defined. Hence, conditional on $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, the joint distribution of $\mathbf{R}_{O_1}$ and $\mathbf{R}_{I'}$ does not depend on the specific realization of $\mathbf{R}_{O_2}$. This implies first that (setting $O_1 = \emptyset$)

$$\Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = \Pr\left\{\mathbf{R}_{I'} \leq \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}.$$

[3] Indeed, by the above lemma, $\Pr\{\mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ does not depend on $\boldsymbol{\ell}_{O_2}$ as long as it is strictly positive. In addition, provided that the conditional distribution is well-defined (i.e., $\boldsymbol{\ell}_{O_2} \gg \mathbf{r}_{O_2}$), $\Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'}, \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is equal to 0 if $\ell_o < r_o$ for some $o \in O_1$. In this case, this remains equal to 0 irrespective of $\boldsymbol{\ell}_{O_2}$. Finally, if $\boldsymbol{\ell}_{O_1} \gg \mathbf{r}_{O_1}$ then the above lemma implies that $\Pr\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1}, \mathbf{R}_{I'} = \boldsymbol{\ell}_{I'}, \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\}$ is a strictly positive number which does not depend on $\boldsymbol{\ell}_{O_2}$.

Next, using Equation (S1) with $I' = \emptyset$, we also obtain that

$$\Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}_{O_2} = \boldsymbol{\ell}_{O_2}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}. \tag{S2}$$

Now, pick an arbitrary $o \in O_1$. We have

$$\begin{aligned}
& \Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\
= \; & \Pr\left\{\mathbf{R}_{O_1\setminus\{o\}} = \boldsymbol{\ell}_{O_1} \mid R_o = \ell_o, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \Pr\left\{R_o = \ell_o \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\
= \; & \Pr\left\{\mathbf{R}_{O_1\setminus\{o\}} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \Pr\left\{R_o = \ell_o \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\},
\end{aligned}$$

where the last equality comes from Equation (S2) above. Now, applying the argument inductively, we obtain

$$\Pr\left\{\mathbf{R}_{O_1} = \boldsymbol{\ell}_{O_1} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = \prod_{o \in O_1} \Pr\left\{R_o = \ell_o \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}.$$

In other words, conditional on $\mathbf{R}^*_{O'} = \mathbf{r}_{O'}$ and $\mathcal{E}$, $\{R_o\}_{o\in O'}$ is a collection of mutually independent random variables (not necessarily identically distributed). In addition, conditional on $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, for each $o \in O'$, R_o is stochastically dominated by the uniform distribution over $\{\lceil \log^{1+\varepsilon}(n)\rceil + 1, ..., n\}$. Indeed, the above lemma implies that for any $o \in O'$, $\Pr\left\{R_o = \ell_o \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = 0$ if $\ell_o \le r_o$ and is constant over all possible ℓ_o such that $\ell_o > r_o$. Thus, in the latter case, $\Pr\left\{R_o = \ell_o \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} = \frac{1}{n - r_o}$. In other words, given $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, for $o \in O'$, R_o follows a uniform distribution over $\{r_o + 1, ..., n\}$. Since $o \in O' \subseteq \bar{O} \subseteq \tilde{O}$, we must have $r_o < \log^{1+\varepsilon}(n)$ and so R_o is stochastically dominated by the uniform distribution over $\{\lceil \log^{1+\varepsilon}(n)\rceil + 1, ..., n\}$. To recap, conditional on $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, $\{R_o\}_{o\in O'}$ is a collection of independent random variables that is stochastically dominated by a collection of $|O'|$ iid random variables distributed according to a uniform distribution over $\{\lceil \log^{1+\varepsilon}(n)\rceil + 1, ..., n\}$, i.e.,

$$\begin{aligned}
\Pr\left\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} &= \prod_{o\in O'} \Pr\left\{R_o \le \ell_o \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\
&\ge \prod_{o\in O'} \Pr\left\{Y_o \le \ell_o\right\}.
\end{aligned} \tag{S3}$$

Now, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,

$$\begin{aligned}
& \Pr\left\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\
= \; & \Pr\left\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \Pr\left\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\} \\
\ge \; & \prod_{o\in O'} \Pr\left\{Y_o \le \ell_o\right\} \Pr\left\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\right\}.
\end{aligned}$$

where the inequality comes from the Equation (S3) together with the fact that the distribution of $\mathbf{R}_{I'}$ does not depend on the specific realization of $\mathbf{R}_{O'}$, as we already claimed. Hence, we obtain

$$\begin{aligned}
& \Pr\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathcal{E}\} \\
= & \sum_{\mathbf{r}_{O'}} \Pr\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'} \mid \mathcal{E}\} \Pr\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\
\ge & \sum_{\mathbf{r}_{O'}} \Pr\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'} \mid \mathcal{E}\} \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \Pr\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\
= & \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \sum_{\mathbf{r}_{O'}} \Pr\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'} \mid \mathcal{E}\} \Pr\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathbf{R}^*_{O'} = \mathbf{r}_{O'}, \mathcal{E}\} \\
= & \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \Pr\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid \mathcal{E}\}
\end{aligned}$$

as claimed.

Note further that, conditional on $\{\mathbf{R}^*_{O'} = \mathbf{r}_{O'}\}$ and $\mathcal{E}$, $\{R_o\}_{o \in O'}$ stochastically dominates the collection of $|O'|$ iid random variables $U_1, ..., U_{|O'|}$ where the distribution of U_o is uniform over $\{1, ..., n\}$. Using a similar argument as above, we obtain that, conditional on $\mathcal{E}$, $\{R_o\}_{o \in O'}$ stochastically dominates a collection of $|O'|$ iid random variables distributed according to a uniform distribution over $\{1, ..., n\}$ and we can easily complete the proof of the second part of Proposition S2.

Finally, to complete the proof of part (a) of Theorem 2, recall that $\frac{|\bar{O}|}{n} \xrightarrow{p} 1$. Hence, for any given integer K (which does not depend on n), $\Pr\left\{\{o_1, ..., o_K\} \subseteq \bar{O}\right\}$ converges to 1. In addition, from the above proposition, we directly obtain that for any $I' \subseteq I$ and for any $O' \subseteq O$, for any $\boldsymbol{\ell}_{O'}, \boldsymbol{\ell}_{I'}$,

$$\Pr\left\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\} \ge \prod_{o \in O'} \Pr\{Y_o \le \ell_o\} \times \Pr\left\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\}$$

and, in addition,

$$\Pr\left\{\mathbf{R}_{O'} \le \boldsymbol{\ell}_{O'}, \mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\} \le \prod_{o \in O'} \Pr\{U_o \le \ell_o\} \times \Pr\left\{\mathbf{R}_{I'} \le \boldsymbol{\ell}_{I'} \mid O' \subseteq \bar{O}\right\}.$$

S.6.2 Proof of Theorem 2 (b)

Fix $x \in [0,1]$. By the above result, given $\{\bar{O} = O''\}$, the collection $\{\mathbf{1}_{\{\bar{R}_o \le x\}}\}_{o \in \bar{O}}$ is stochastically dominated by $\{\mathbf{1}_{\{\bar{Y}_o \le x\}}\}_{o \in \bar{O}}$ where $\bar{Y}_o$ is $\frac{1}{n} U\{\lceil \log^{1+\varepsilon}(n) \rceil + 1, ..., n\}$ which converges in distribution to $U[0,1]$. Similarly, given $\{\bar{O} = O''\}$, the collection $\{\mathbf{1}_{\{R_o \le x\}}\}_{o \in \bar{O}}$ stochastically dominates the collection $\{\mathbf{1}_{\{\bar{U}_o \le x\}}\}_{o \in \bar{O}}$ where $\bar{U}_o$ is $\frac{1}{n} U\{1, ..., n\}$ which converges in distribution to $U[0,1]$.

Now, fix any $\delta > 0$ and let us further condition w.r.t. the event that $|\bar{O}| \geq (1-\delta)n$. Note that the probability of this event goes to 1 as n grows. Now, conditional on $|\bar{O}| \geq (1-\delta)n$ and $\{\bar{O} = O''\}$, we have,

$$\begin{aligned}
\frac{1}{n}\sum_{o\in O}\mathbf{1}_{\{\bar{R}_o\leq x\}} &= \frac{1}{n}\left(\sum_{o\in\bar{O}}\mathbf{1}_{\{\bar{R}_o\leq x\}} + \sum_{o\in O\setminus\bar{O}}\mathbf{1}_{\{\bar{R}_o\leq x\}}\right) \\
&\leq_{st} \frac{|\bar{O}|}{n}\frac{1}{|\bar{O}|}\sum_{o\in\bar{O}}\mathbf{1}_{\{\bar{R}_o\leq x\}} + \frac{|O\setminus\bar{O}|}{n} \\
&\leq_{st} (1-\delta)\frac{1}{|\bar{O}|}\sum_{o\in\bar{O}}\mathbf{1}_{\{\bar{R}_o\leq x\}} + \delta \\
&\leq_{st} (1-\delta)\frac{1}{|\bar{O}|}\sum_{o\in\bar{O}}\mathbf{1}_{\{\bar{Y}_o\leq x\}} + \delta \xrightarrow{p} (1-\delta)x + \delta
\end{aligned}$$

where the convergence result is by the LLN. Similarly, we must have that conditional on the above events,

$$\frac{1}{n}\sum_{o\in O}\mathbf{1}_{\{\bar{R}_o\leq x\}} \geq_{st} (1-\delta)\frac{1}{|\bar{O}|}\sum_{o\in\bar{O}}\mathbf{1}_{\{\bar{U}_o\leq x\}} \xrightarrow{p} (1-\delta)x.$$

Hence, conditional on $|\bar{O}| \geq (1-\delta)n$ and $\{\bar{O} = O''\}$, we must have that with probability going to 1, $\frac{1}{n}\sum_{o\in O}\mathbf{1}_{\{\bar{R}_o\leq x\}}$ falls in $[(1-\delta)x, (1-\delta)x+\delta]$. This must also be true if we only condition w.r.t. $|\bar{O}| \geq (1-\delta)n$. Since $|\bar{O}| \geq (1-\delta)n$ is a large probability event, we must have that, unconditionally, with probability going to 1, $\frac{1}{n}\sum_{o\in O}\mathbf{1}_{\{\bar{R}_o\leq x\}}$ falls in $[(1-\delta)x, (1-\delta)x+\delta]$. Since $\delta > 0$ is arbitrary, this implies that

$$\frac{1}{n}\sum_{o\in O}\mathbf{1}_{\{\bar{R}_o\leq x\}} \xrightarrow{p} x.$$

S.7 Proof of Corollary 1

Denote $\text{Rank}_i(o)$ (resp., $\text{Rank}_o(i)$) for the rank of object o (individual i) in i's preferences (o's priority ordering). Let us denote by E the joint event $\{o \in \bar{O} \text{ and } \text{Rank}_o(i) > R_o^*\}$ and let us first show that

$$\Pr\{R_o > \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} = \Pr\{R_o < \text{ Rank}_o(i)\,|E, R_i > \text{ Rank}_i(o)\} = \frac{1}{2}.$$

Consider the event $\{R_i > \text{Rank}_i(o), R_o > \text{Rank}_o(i), o \in \bar{O}, \text{Rank}_o(i) > R_o^*\}$. Pick any preference profile under which this event is true. Let k be the individual with rank R_o (i.e.,

$\mathrm{TTC}(o) = k$). Since $o \in \bar{O} \subseteq \hat{O}$, $R_o^* < R_o$. In addition, by assumption, we must have $R_o^* < \mathrm{Rank}_o(i)$. Hence, both k and i have a priority ranking at o worse than that of R_o^*. Now, let us swap k and i in object o's priority ordering. Because both k and i have a priority ranking at o worse than that of R_o^*, this has no impact on the outcome of TTC. Thus, we must have $\{R_i > \mathrm{Rank}_i(o), R_o < \mathrm{Rank}_o(i), o \in \bar{O}, \mathrm{Rank}_o(i) > R_o^*\}$. Thus, we have an injection from the set of profiles of preferences and priorities yielding $\{R_i > \mathrm{Rank}_i(o), R_o > \mathrm{Rank}_o(i), o \in \bar{O}, \mathrm{Rank}_o(i) > R_o^*\}$ to the one yielding $\{R_i > \mathrm{Rank}_i(o), R_o < \mathrm{Rank}_o(i), o \in \bar{O}, \mathrm{Rank}_o(i) > R_o^*\}$, showing that

$$\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\} \leq \Pr\left\{R_o < \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\}.$$

Clearly, a symmetric reasoning shows that

$$\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\} \geq \Pr\left\{R_o < \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\}$$

and so we can conclude that

$$\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\} = \Pr\left\{R_o < \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\} = \frac{1}{2}.$$

Using a symmetric argument,

$$\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E, R_i < \ \mathrm{Rank}_i(o)\right\} = \Pr\left\{R_o < \ \mathrm{Rank}_o(i)\,|E, R_i < \ \mathrm{Rank}_i(o)\right\} = \frac{1}{2}.$$

In words, the probability that $R_o > \mathrm{Rank}_o(i)$ does not depend on the realization of event $\{R_i > \mathrm{Rank}_i(o)\}$, and it follows that

$$\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E\right\} = \Pr\left\{R_o < \ \mathrm{Rank}_o(i)\,|E\right\} = \frac{1}{2}.$$

Now, it is enough for our purpose to show that the probability of the joint event $E = \{o \in \bar{O}, \mathrm{Rank}_o(i) > R_o^*\}$ goes to 1. Indeed, we already know that the probability of $\{o \in \bar{O}\}$ goes to 1. In addition, $\mathrm{Rank}_o(i)$ follows a uniform distribution over $\{1, ..., |I|\}$. Hence, $\Pr\{\mathrm{Rank}_o(i) > \log^{1+\varepsilon}(n)\}$ goes to 1. We also know that $\Pr\{R_o^* < \log^{1+\varepsilon}(n)\}$ goes to 1. Thus, $\Pr\{\mathrm{Rank}_o(i) > R_o^*\}$ goes to 1 as well and so $\Pr(E)$ goes to 1. The first statement is thus proven.

For the second statement, let us consider the probability that (i, o) blocks TTC given that event E holds. This is,

$$\begin{aligned}
&\Pr\left\{R_o > \ \mathrm{Rank}_o(i), R_i > \ \mathrm{Rank}_i(o)\,|E\right\} \\
= \ &\Pr\left\{R_i > \ \mathrm{Rank}_i(o)\,|E\right\}\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E, R_i > \ \mathrm{Rank}_i(o)\right\} \\
= \ &\Pr\left\{R_i > \ \mathrm{Rank}_i(o)\,|E\right\}\Pr\left\{R_o > \ \mathrm{Rank}_o(i)\,|E\right\}
\end{aligned}$$

where the last equality holds by our argument above.

Now, we first claim that

$$\left|\Pr\left\{R_o > \text{ Rank}_o(i), R_i > \text{ Rank}_i(o)\right\} - \Pr\left\{R_i > \text{ Rank}_i(o)\right\}\Pr\left\{R_o > \text{ Rank}_o(i)\right\}\right|$$

goes to 0 as the market grows large. We have shown that conditional on E, this difference is just equal to 0. Now, since, as we already showed, $\Pr(E)$ goes to 1, the convergence must hold unconditionally.

Second, we know that $\text{Rank}_o(i)$ is a uniform distribution over $\{1, ..., |I|\}$ and let us observe the realization of $\text{Rank}_o(i)$ has no impact on the distribution of R_o. Now, Proposition S2 showing that $\bar{R}_o$ converges in distribution to $U[0,1]$ gives us that $\Pr\left\{R_o > \text{ Rank}_o(i)\right\}$ goes to $\frac{1}{2}$. Taken together the above two points yield

$$\left|\Pr\left\{R_o > \text{ Rank}_o(i), R_i > \text{ Rank}_i(o)\right\} - \frac{1}{2}\Pr\left\{R_i > \text{ Rank}_i(o)\right\}\right|$$

goes to 0 as the market grows large. This completes the proof of the first part of the statement of Corollary 1 since $\frac{1}{2}\Pr\left\{R_i > \text{ Rank}_i(o)\right\}$ is equal to the probability that (i, o) blocks RSD (recall the equivalence result by Carroll (2014)).

Finally, we can easily obtain that the difference between the expected fractions of blocking pairs under TTC and that under RSD converges to 0.

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