Recommender Systems as Incentives for Social Learning

Supplementary Material

Yeon-Koo Che* Johannes Hörner†

July 6, 2017

*Department of Economics, Columbia University, 420 West 118th Street, 1029IAB, New York, NY 10027, USA. Email: yeonkooche@gmail.com.

†Yale University, 30 Hillhouse Ave., New Haven, CT 06520, USA, and Toulouse School of Economics, 21 Allée de Brienne, 31015 Toulouse, France, Email: joh.horner@gmail.com.

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C.1 Proof of Lemma 1

We provide a C^1 solution to the HJB equation of the planner, which by verification implies that this solution is equal to the value function, and the associated policy an optimal policy. There are two cases to consider. For completeness, the case $c \in [1/2, 1]$ is covered. As before, we work with $\ell := p/(1-p)$ and $k := c/(1-c)$,

1. Case 1: $rk > \sqrt{m_a + r}\sqrt{m_b + r}$. This case is a straightforward extension of the baseline model, in which "on each half" of the unit interval, we have experimentation with only one type of agent. More specifically, let

$$\ell_b := \frac{r}{m_\beta + r}k, \qquad \ell_a := \frac{m_\alpha + r}{rk},$$

where as before $k = c/(1-c)$. Note that $rk > \sqrt{m_a + r}\sqrt{m_b + r}$ is equivalent to $\ell_a < \ell_b$. and the optimal policy is given by, in terms of $\ell = p/(1-p)$,

$$(\alpha_a, \alpha_b) = \begin{cases} (1, 0), & \text{for } \ell \le \ell_a, \\ (0, 0), & \text{for } \ell \in (\ell_a, \ell_b) \\ (0, 1), & \text{for } \ell \ge \ell_b. \end{cases}$$

As for optimality, consider (wlog) the case $\ell \ge \sqrt{\ell_a \ell_b}$. Then the value function

$$V(\ell) = \begin{cases} 0, & \text{for } \ell \in [1, \ell_b), \\ \frac{m_b\left(\ell_b\left(m_b\left(\frac{\ell}{\ell_b}\right)^{-\frac{r}{m_b}} - m_b - r\right) + r\ell\right)}{(\ell+1)(\ell_b(m_b+r)+r)}, & \text{for } \ell \ge \ell_b, \end{cases}$$

is a C^1 function that alongside the candidate policy solves the HJB equation

$$\begin{aligned} rV(\ell) \;&=\; \max_{\alpha_a, \alpha_b} \{r(m_b\alpha_b(p-c) + m_a\alpha_a(1-p-c)) + pm_b\alpha_b(m_b(1-c) - V(\ell)) \\ &+\; (1-p)m_a\alpha_a(m_a(1-c) - V(\ell)) + \ell(\alpha_a m_a - \alpha_b m_b)V'(\ell). \end{aligned}$$

2. Case 2: $rk \le \sqrt{m_a + r}\sqrt{m_b + r}$. The policy now involves two thresholds, $\underline{\ell}, \bar{\ell}$, with

$0 < \underline{\ell} \leq \bar{\ell}$ such that

$$(\alpha_a, \alpha_b) = \begin{cases} (1,0), & \text{for } \ell < \underline{\ell}, \\ (1, m_a/m_b), & \text{for } \ell = \underline{\ell}, \\ (1,1), & \text{for } \ell \in (\underline{\ell}, \bar{\ell}) \\ (0,1), & \text{for } \ell \geq \bar{\ell}. \end{cases}$$

Again, the proof is by verification. We show that $\underline{\ell}, \bar{\ell}$ exists such that the resulting payoff function is C^1 and solves the HJB. In fact, we define

$$\underline{\ell} := \frac{k\,(m_a + r)}{km_a + m_b + r}.$$

as well as $\bar{\ell}$ as the root $\ell \geq \underline{\ell}$ of

$$\left(\ell^{-\frac{m_a}{m_b - m_a}} + \ell^{-\frac{m_b}{m_b - m_a}}\right) \underline{\ell}^{\frac{m_a}{m_b - m_a}} (\ell/\underline{\ell})^{-\frac{r}{m_b - m_a}} = \frac{((m_a + r)(1 - k\ell) + km_b)\,(1 + \ell)}{k\,(m_b - m_a)\,\ell}, \tag{1}$$

which can be readily shown to exist and be unique given $rk \leq \sqrt{m_a + r}\sqrt{m_b + r}$. We define

$$V_1(\ell) = \frac{k\,(m_a + r(1 + \ell))\left(m_a \ell^{\frac{r}{m_a}} \underline{\ell}^{-\frac{m_a + r}{m_a}} - \frac{m_a + r}{\underline{\ell}} + \frac{r}{\ell}\right)}{(k+1)\,(m_a + r)},$$

$$V_2(\ell) = \frac{rm_a\left((\ell + 1)\,((k\ell - 1)\,(m_a + r) - km_b) + k\,(m_b - m_a)\left(\ell^{\frac{m_a}{m_a - m_b}} + \ell^{\frac{m_b}{m_a - m_b}}\right)\ell^{\frac{r}{m_a - m_b} + 1}\underline{\ell}^{\frac{m_a + r}{m_b - m_a}}\right)}{(\ell + 1)^2\,(m_a + r)},$$

and

$$V_3(\ell) = \frac{(\ell/\bar{\ell})^{-\frac{r}{m_b}}\left(m_b(k - \bar{\ell}) + (k+1)(\bar{\ell} + 1)V_2(\bar{\ell})\right) + m_b(\ell - k)}{(k+1)(\ell + 1)}.$$

We finally set

$$V(\ell) = \begin{cases} V_1(\ell), & \text{for } \ell \leq \underline{\ell}, \\ V_2(\ell), & \text{for } \ell \in (\underline{\ell}, \bar{\ell}), \\ V_3(\ell), & \text{for } \ell \geq \bar{\ell}, \end{cases}$$

The formulas for V_j, $j = 1, 2, 3$ are precisely those that result from the HJB equation evaluated at $\alpha = (1,0), (1,1)$ and $(0,1)$, respectively, taking as given $\underline{\ell}, \bar{\ell}$. The value of $\underline{\ell}$ has been picked so that $V_1 - V_2$ is C^1 at $\underline{\ell}$, and that the coefficient on α_b of the right-hand side of equation (1) is zero at $\underline{\ell}$ (as it happens, both conditions are satisfied simultaneously at that value), while the choice of $\bar{\ell}$ has been chosen so that the coefficient on α_b of the right-hand side of equation (1) is zero at $\bar{\ell}$ (here as well, it also follows that $V_1 - V_2$ is C^1 at $\bar{\ell}$). It is readily verified that these coefficients have the right variations as a function of ℓ, given our candidate V, namely, the coefficient on α_b (α_a) is increasing (decreasing) in ℓ, ensuring that the HJB equation is satisfied.

Finally, we note that, by definition $\underline{\ell} < k$, so that $\underline{p} = \underline{\ell}/(1+\underline{\ell}) < c$. On the other hand, equation (1) is equivalent to $h(\ell) = 0$, where

$$h(\ell) = \frac{(\ell+1)\left(m_a(k\ell-1) - km_b + r(k\ell-1)\right) + k\left(m_b - m_a\right)\left(\ell^{\frac{m_a}{m_a-m_b}} + \ell^{\frac{m_b}{m_a-m_b}}\right)\underline{\ell}^{-\frac{m_a+r}{m_a-m_b}}\ell^{\frac{r}{m_a-m_b}+1}}{k+1}.$$

Without loss, we scale m_a, m_b, so that $m_a + m_b = 1$, and we evaluate at $\ell = k^{-1}$. This gives

$$h(k^{-1}) = (1-2m_a)\left(\frac{k^2\left(m_a+r\right)}{(k-1)m_a + r + 1}\right)^{\frac{m_a+r}{1-2m_a}} + m_a - 1,$$

Given that $m_a < 1/2$, this expression is increasing in k, so evaluating at $k = 1$ (restricting attention to $c \leq 1/2$), we obtain as upper bound,

$$(1-2m_a)\left(\frac{m_a+r}{r+1}\right)^{\frac{m_a+r}{1-2m_a}} + m_a - 1,$$

an expression that is convex in x under the change of variable $r \mapsto (1-2m_a)x - m_a$, yet negative whether $x = m_a/(1-2m_a)$ (the lower bound on x when $r = 0$) as well as negative when $x \to \infty$. Hence, $h(k^{-1}) < 0$, yet also $\lim_{\ell\to\infty} h(\ell) = +\infty$, so that h admits a root in (k^{-1}, ∞). Finally, we note that $h(\ell)/(1+\ell)$ is convex. Indeed, taking second derivatives yields

$$\frac{k\left(1+r-m_a\right)\left(m_a+r\right)\underline{\ell}^{\frac{m_a+r}{2m_a-1}}\ell^{\frac{r+2-3m_a}{1-2m_a}}}{1-2m_a} > 0,$$

and so, on the other hand, h admits at most one root. Hence, it admits exactly one root, and $\bar{\ell} > k^{-1}$, that is, $\bar{p} > 1 - c$, for $c < 1/2$. $\square$

C.2 Proof of Proposition 5

Note that if $p^0 \in [c, 1-c]$, or $p^0 \in \{0, 1\}$, the first-best is possible. Hence, fix $p^0 \in (0,1) \setminus [c, 1-c]$, and as usual $\ell^0 = p^0/(1-p^0)$. First, consider, $p^0 < c$. Let $\underline{p} := \inf\{p \geq p^0 : \alpha_b(p) > 0\}$. Let $\{p_t\}_t$ denote the trajectory under the optimal (second-best) policy, and define $\hat{t} := \inf\{t : p_t \geq \underline{p}\}$. If $\hat{t} = +\infty$, then consumers b never experiment. Hence, the solution must coincide with the baseline solution with one type of consumers only (here, the a-types). If instead, $\hat{t} < +\infty$, then the first-best is possible once time $\hat{t}$ is reached, and we can consider the finite-horizon problem with only a-types experimenting (in some proportion) until time $\hat{t}$, at which point the designer receives the first-best value. Because in either case, only a-agents experiment (if at all) until $\hat{t}$, this reduced problem is not constrained by incentives, and it readily follows that the solution is bang-bang. In case $\hat{t} < +\infty$, it must be that $\underline{p} = c$ (it cannot be lower than c, for b-agents to be willing to experiment, and first-best is available at that value, so the designer cannot do better once this value is reached). Thus, we are led to a simple comparison of values. Note that, from the proof of Lemma 2, case 1, experimentation with only a-type agents would stop at $\ell_a = \frac{m_\alpha + r}{rk}$. Yet

$$\frac{m_\alpha + r}{rk} > k \iff k \leq \sqrt{1 + \frac{m_\alpha}{r}},$$

which is necessarily the case for $c < 1/2$ (or $k < 1$). Hence, the optimal policy must be to experiment with a-types until $\ell = k$ is reached, and then switch to the optimal policy.

The reasoning is identical in case $p^0 > 1 - c$, noting that, here as well, the threshold where experimentation with only b-types would stop, $\ell_b = \frac{r}{r + m_\beta} k$ is smaller than the threshold at which agent a are willing to experiment, $\ell = k^{-1}$ (that is, $p = 1 - c$). Hence, the optimal policy involves experimenting with b-types only until p is reached, at which point the first-best policy is followed.

As noted in the discussion following Proposition 5, if $c > 1/2$, it might be the case that it is better to simply experiment with one type of agent (*e.g.*, in case $p^0 < c$ and $k \leq \sqrt{1 + \frac{m_\alpha}{r}}$, it might be that the value of stopping experimentation at ℓ_a yields a higher value than experimenting with a-types until $\ell = k$ (and getting first-best then). $\square$

D Proofs for the Results from Section 6

D.1 Proof of Proposition 4 (from Section 6.1)

The objective function reads

$$\int_{t \geq 0} e^{-rt} \left(g_t(1 - \bar{c}) + (1 - g_t - b_t)(q_H \alpha_H (p_t - c_L) + q_L \alpha_L (p_t - c_L)) \right) \mathrm{d}t,$$

where $\bar{c} := q_H c_H + q_L c_L$. Substituting for g_t, b_t and re-arranging, this gives

$$\int_{t \geq 0} e^{-rt} \ell(t) \left(\alpha_H(t) q_H \left(1 - c_H \left(1 + \frac{1}{\ell(t)} \right) \right) + \alpha_L(t) q_L \left(1 - c_L \left(1 + \frac{1}{\ell(t)} \right) \right) - (1 - \bar{c}) \right) \mathrm{d}t.$$

As before, it is more convenient to work with $t(\ell)$ as the state variable, and doing the change of variables gives

$$\int_0^{\ell_0} e^{-rt(\ell)} \left(x_H(\ell)u_H(\ell) + x_L(\ell)u_L(\ell) - \frac{1-\bar{c}}{\rho} \right) \mathrm{d}\ell,$$

where for $j = L, H$, $x_j(\ell) := 1 - c_j \left(1 + \frac{1}{\ell}\right) + \frac{1-\bar{c}}{\rho}$, and $u_j(\ell) := \frac{q_j \alpha_j(t(\ell))}{\rho + q_L \alpha_L(t(\ell)) + q_H \alpha_H(t(\ell))}$ are the control variables that take values in the sets $\mathcal{U}^j(\ell) = [\underline{u}_k, \bar{u}_k]$ (whose definition depends on first- vs. second-best). This is to be maximized subject to

$$t'(\ell) = \frac{u_H(\ell) + u_L(\ell) - 1}{\rho \lambda \ell}.$$

As before, we invoke Pontryagin's principle. There exists an absolutely continuous function $\eta : [0, \ell_0] \to \mathbf{R}$, such that, a.e.,

$$\eta'(\ell) = re^{-rt(\ell)} \left(x_H(\ell)u_H(\ell) + x_L(\ell)u_L(\ell) - \frac{1-\bar{c}}{\rho} \right),$$

and u_j is maximum or minimum, depending on the sign of

$$\phi_j(\ell) := \rho \lambda \ell e^{-rt(\ell)} x_j(\ell) + \eta(\ell).$$

This is because this expression cannot be zero except for a specific value of $\ell = \ell_j$. Namely, note first that, because $x_H(\ell) < x_L(\ell)$ for all ℓ, at least one of $u_L(\ell), u_H(\ell)$ must be extremal, for all ℓ. Second, upon differentiation,

$$\phi'_H(\ell) = e^{-rt(\ell)} \left(\left(\lambda - \frac{r}{\rho} \right) (1 - \bar{c}) + \rho\lambda(1 - c_H) + ru_L(\ell)(c_H - c_L) \left(1 + \frac{1}{\ell} \right) \right)$$

implies that, if $\phi_H(\ell) = 0$ were identically zero over some interval, then $u_L(\ell)$ would be extremal over this range, yielding a contradiction, as the right-hand side cannot be zero identically, for $u_L(\ell) = \bar{u}_L(\ell)$. Similar reasoning applies to $u_L(\ell)$, considering $\phi'_L(\ell)$. Hence, the optimal policy is characterized by two thresholds, ℓ_H, ℓ_L, with $\ell_0 \geq \ell_H \geq \ell_L \geq 0$, such that both types of regular consumers are asked to experiment whenever $\ell \in [\ell_H, \ell_0]$, low-cost consumers are asked to do so whenever $\ell \in [\ell_L, \ell_0]$, and neither is asked to otherwise.

We now characterize the threshold beliefs under first-best and second-best policies. Throughout, we shall use superscript $**$ to denote the first-best and superscript $*$ to denote the second-best policy. By the principle of optimality, the threshold ℓ_L must coincide with $\ell^* = \ell^{**}$ in the case of only one type of regular consumers (with cost c_L). To compare ℓ_H^*

and ℓ_H^{**}, we proceed as in the bad news case, by noting that, in either case,

$$\phi_H(\ell_H) = 0,$$

and

$$\phi_H(\ell_L) = \phi_L(\ell_L) + \rho\lambda\ell_L e^{-rt(\ell_L)}(x_H(\ell_L) - x_L(\ell_L)) = -\rho\lambda e^{-rt(\ell_L)}(c_H - c_L)\,(1+\ell_L)\,.$$

Hence,

$$\int_{\ell_L}^{\ell_H} e^{rt(\ell_L)}\phi_H'(\ell)\mathrm{d}\ell = \rho\lambda(c_H - c_L)\,(1+\ell_L)$$

holds both for the first- and second-best. Note now that, in the range $[\ell_L, \ell_H]$,

$$e^{rt(\ell_L)}\phi_H'(\ell) = e^{-r\int_{\ell_H}^{\ell}\frac{u_L(l)+u_H(l)-1}{\rho\lambda l}\mathrm{d}l}\left(\left(\lambda - \frac{r}{\rho}\right)(1-\bar{c}) + \rho\lambda(1-c_H) + ru_L(\ell)(c_H - c_L)\left(1+\frac{1}{\ell}\right)\right).$$

Because $\bar{\alpha}_L(\ell) > \bar{\alpha}_H(\ell)$, $\bar{u}_L^*(\ell) > \bar{u}_L^{**}(\ell)$, and also $\bar{u}_L^{**}(\ell) + \bar{u}_H^{**}(\ell) \geq \bar{u}_L^*(\ell) + \bar{u}_H^*(\ell)$, so that, for all ℓ in the relevant range,

$$e^{rt(\ell_L)}\frac{\mathrm{d}\phi_H^{**}(\ell)}{\mathrm{d}\ell} < e^{rt(\ell_L)}\frac{\mathrm{d}\phi_H^{*}(\ell)}{\mathrm{d}\ell},$$

and it then follows that $\ell_H^* < \ell_H^{**}$. □

D.2 Proof of Proposition 5 (from Section 6.2)

Let us posit that the candidate equilibrium is the first-best policy. Hence, the agent who arrives at some random time is being told to experiment if and only if the posterior of the designer is above ℓ^* at that moment. Plainly, if she is told not to consume, she will gladly abide. On the other hand, conditional on being told to consume, she will form a posterior belief on the time t that might prevail –call the corresponding cdf F– and compute her expected utility as follows

$$\mathcal{U} = \int_{t\geq 0}\frac{\Pr[\tilde{p}_t = p_t]\alpha_t p_t + (1-\Pr[\tilde{p}_t = p_t])\cdot 1}{\Pr[\tilde{p}_t = p_t]\alpha_t + (1-\Pr[\tilde{p}_t = p_t])}\mathrm{d}F(t),$$

which is as before (here, $\tilde{p}_t$ is the random posterior belief of the principal), except that now α_t is either 0 or 1. Explicitly, if $t < t^*$, the time at which $p_t = p^*$ conditional on no news, then $\alpha_t = 1$ and the first term is simply p^0; if $t \geq t^*$, then $\alpha = 0$ and the first term is 1. Hence,

$$\mathcal{U} = p^0 F(t^*) + 1 - F(t^*)$$

and so incentive compatibility, which requires that $\mathcal{U} \geq c$, is satisfied if, and only if,

$$F(t^*) \leq \frac{1-c}{1-p^0} = \frac{1+\ell^0}{1+k}. \tag{2}$$

It remains to determine the cdf F. Note that the agent assigns probability 1 to being told to consume when $t < t^*$, so being told to do so skews her belief towards such times. Explicitly, for $t < t^*$,

$$\begin{aligned} \mathrm{d}F(t) &= \frac{\Pr[t \text{ \& ``buy''}]}{\Pr[\text{``buy''}]} \\ &= \frac{\int_{j \geq t} \frac{1}{j} \xi e^{-\xi j} \mathrm{d}j}{\int_{j=0}^{t^*} \xi e^{-\xi j} \mathrm{d}j + \int_{s \geq t^*} \int_{j=t^*}^{s} \left(\frac{p^0}{s} \left(1 - e^{-\lambda(1+\rho)t^* - \lambda\rho(j-t^*)}\right) \mathrm{d}j \right) \xi e^{-\xi s} \mathrm{d}s}. \end{aligned} \tag{3}$$

Integrating (3), we obtain

$$F(t^*) = \frac{\xi t^* E_1(t^*\xi) + 1 - e^{-\xi t^*}}{\frac{\xi p^0 \left(e^{-\lambda t^*} E_1(t^*(\xi+\lambda\rho)) + \mathrm{Ei}(-t^*\xi)\left(\lambda\rho t^* + e^{-\lambda(\rho+1)t^*}\right)\right)}{\lambda\rho} + (p^0-1)e^{-\xi t^*} + 1}, \tag{4}$$

where $\mathrm{Ei}(x), E_1(x)$ are the exponential integral functions $-\int_{-x}^{\infty} \frac{e^{-t}}{t} \mathrm{d}t$, and $\int_{x}^{\infty} \frac{e^{-t}}{t} \mathrm{d}t$. We recall the definition of t^*, namely, $\ell^0 e^{-\lambda(1+\rho)t^*} = \ell^*$, or $t^* = \frac{1}{\lambda(1+\rho)} \ln \frac{\ell^0}{\ell^*}$. Hence, we can plug equation (4) into (2) from the paper to get a condition on the parameters. Taking Taylor expansions in ξ gives that the condition is satisfied when ξ is small enough.

D.3 Proof of Proposition 6 (from Section 6.3)

Since the designer can induce at most a fraction $\hat{e}(\ell_t)$ of the agents to explore (with a slight abuse of notation), she can attain at most the value:

$$[SB - Naive] \qquad \sup_{\alpha} \int_{t \geq 0} e^{-rt} \left(\ell^0 - \ell_t - \alpha_t (k - \ell_t)\right) \mathrm{d}t$$

subject to

$$\dot{\ell}_t = -\lambda \alpha_t \ell_t, \; \forall t, \text{ and } \ell_0 = \ell^0, \tag{5}$$

$$0 \leq \alpha_t \leq \hat{e}(\ell_t), \; \forall t. \tag{6}$$

This problem is the same as $[SB]$, except that $\rho = 0$ and that the designer can induce any measure $\alpha_t \in [0, \hat{e}(\ell_t)]$ of agents to experiment at each time t. Although this latter constraint is ostensively a relaxation of the true constraint, one can show that any $\alpha_t \in [0, \hat{e}(\ell_t)]$ can be attained by the designer. Any $\alpha_t \leq \bar{\alpha}(\ell_t)$ is attained by simply spamming to a fraction α_t of randomly selected agents, which the rational agents find credible. Any $\alpha_t \in (\bar{\alpha}(\ell_t), \rho_n]$, if it is well-defined, can be achieved by "blasting" spam to a fraction α_t/ρ_n, which only naive agents will follow and rational agents will ignore. It therefore follows that $[SB - Naive]$ describes the exact problem facing the designer, and can be solved exactly same as $[SB]$. □

D.4 Formulation of the Problem for Section 6.4

The problem is now written as:

$$\int_0^\infty e^{-rt} \left[g_t(1-c) + (1-g_t)\alpha_t(p_t - c) - c(\rho_t)\right] \mathrm{d}t,$$

where

$$g_t = \frac{p^0 - p_t}{1 - p_t} = \frac{\ell^0 - \ell_t}{1 + \ell_0},$$

and

$$\alpha_t \leq \frac{\ell^0 - \ell_t}{k - \ell_t},$$

(ignoring the uninteresting case $k < \ell^0$, in which this constraint can be ignored in an initial phase), $c(\rho) = \rho^2$ is the flow cost of choosing ρ, as well as

$$\dot{\ell}_t = -\lambda(\rho_t + \alpha_t)\ell_t.$$

(Recall that $k := c/(1+c), \ell := p/(1+p)$.)

Rearranging this objective, and ignoring irrelevant constants, the program is equivalent to solving:

$$\max_{\alpha,\rho} \int_0^\infty e^{-rt} \left[\ell^0 - \ell_t - \alpha_t(k - \ell_t) - \rho_t^2(1+\ell^0)(1+k)\right] \mathrm{d}t$$

such that

$$\dot{\ell}_t = -\lambda(\rho_t + \alpha_t)\ell_t, \quad \ell_t = \ell^0, \tag{7}$$

as well as

$$\ell^0 - \ell_t - \alpha_t(k - \ell_t) \geq 0. \tag{8}$$

Here, k and ℓ^0 are constants such that $k > \ell^0 > 0$. T

Here, we prove that the optimal learning of experimentation, α, remains extremal, as in the baseline model. We use Pontryagin's maximum principle, with controls α, ρ, and state ℓ. We define

$$H(\ell, \alpha, \rho, \mu, t) = e^{-rt}\left[\ell^0 - \ell_t - \alpha_t(k - \ell_t) - \rho_t^2(1 + \ell^0)(1 + k)\right] - \mu_t\lambda(\rho_t + \alpha_t)\ell_t,$$

where μ is the costate variable associated with equation (7), and

$$\begin{aligned} L(\ell, \alpha, \rho, \mu, q^1, q^2, t) &= e^{-rt}\left[\ell^0 - \ell_t - \alpha_t(k - \ell_t) - \rho_t^2(1 + \ell^0)(1 + k)\right] \\ &- \mu_t\lambda(\rho_t + \alpha_t)\ell_t + q_t^1(\ell^0 - \ell_t - \alpha_t(k - \ell_t)) + q_t^2\alpha_t. \end{aligned}$$

with q_t^1 the Lagrangian associated with equation (8), and q_t^2 with $\alpha \geq 0$. For notational simplicity, we have ignored the third constraint, $\rho \geq 0$. [As explained above, this constraint *does* bind, but only for large values of t. The argument that follows does not rely on this.]

We have, from the maximum principle, the following optimality conditions:

$$q_t^1 \geq 0, = 0 \text{ if } \ell^0 - \ell_t - \alpha_t(k - \ell_t) > 0, \tag{9}$$

$$q_t^2 \geq 0, = 0 \text{ if } \alpha_t > 0, \tag{10}$$

$$\dot{\mu}_t = -\frac{\partial L(\ell, \alpha, \rho, \mu, q^1, q^2, t)}{\partial \ell} = e^{-rt}(1 - \alpha_t) + \mu_t\lambda(\rho_t + \alpha_t) + q_t^1(1 - \alpha_t). \tag{11}$$

and finally, it must be that (α_t, ρ_t) maximize $L(\ell, \alpha, \rho, \mu, q^1, q^2, t)$ (along the optimal trajectory ℓ). That is, when ρ is interior, taking first-order conditions in $L(\cdot)$ w.r.t. ρ_t,

$$2e^{-rt}\rho_t(1 + \ell_t)(1 + k) + \mu_t\lambda\ell_t = 0. \tag{12}$$

Also, considering that $L(\cdot)$ is linear in α_t, either

$$\alpha_t \in \left\{0, \frac{\ell^0 - \ell_t}{k - \ell_t}\right\}, \tag{13}$$

or

$$e^{-rt}(k - \ell_t) + \mu_t\lambda\ell_t = 0. \tag{14}$$

More generally,

$$e^{-rt}(k - \ell_t) + \mu_t\lambda\ell_t + q_t^1(k - \ell_t) - q_t^2 = 0. \tag{15}$$

First, we consider the case in which $\alpha \notin \left\{0, \frac{\ell^0 - \ell_t}{k - \ell_t}\right\}$, with the purpose of ruling it out. If so, from equation (9) and equation (11), we have

$$\dot{\mu}_t = e^{-rt}(1 - \alpha_t) + \mu_t\lambda(\rho_t + \alpha_t),$$

and from equation (14),

$$\mu_t = -\frac{e^{-rt}(k-\ell_t)}{\lambda \ell_t}.$$

Differentiating, and plugging in the previous equation (using $\dot{\ell}$), we obtain

$$kr = [r + \lambda(1+\rho_t)]\,\ell_t. \tag{16}$$

At the same time, we may combine equation (12) with equation (14) to get

$$\rho_t = \frac{k-\ell_t}{2(1+\ell_t)(1+k)}. \tag{17}$$

Plugging in equation (17) in equation (16), we obtain a unique value of ℓ_t. Hence, we cannot have $\alpha \notin \left\{0, \frac{\ell^0-\ell_t}{k-\ell_t}\right\}$ over an interval. This proves that α must be extremal.

Next, it is possible to derive first-order differential equations for ρ_t on each interval, but these admit no closed-form solution. Hence, we solve the problem numerically for a generic set of parameters, featured in Figure 7 of the paper.

D.5 Proof of Proposition 7 (from Section 6.5)

Here, we extend our model to allow for both good news and bad news and establish among others Proposition 7 in the paper. Specifically, if a flow of size μ consumes the good over some time interval $[t, t+\mathrm{d}t)$, then the designer learns during this time interval that the movie is "good" with probability $\lambda_g(\rho+\mu)\mathrm{d}t$, that it is "bad" with probability $\lambda_b(\rho+\mu)\mathrm{d}t$, where $\lambda_g, \lambda_b \geq 0$, and ρ is the rate of background learning.

The designer commits to the following policy: At time t, she recommends the movie to a fraction $\gamma_t \in [0,1]$ of agents if she learns the movie to be good, a fraction $\beta_t \in [0,1]$ if she learns it to be bad, and she recommends to fraction $\alpha_t \in [0,1]$ if no news has arrived by t. Clearly,

$$\mu_t = \rho + \alpha_t.$$

The designer's belief evolves according to

$$\dot{p}_t = -(\lambda_g - \lambda_b)\mu_t p_t(1-p_t), \tag{18}$$

with the initial value $p_0 = p^0$. It is worth noting that the evolution of the posterior depends on the relative arrival rates of the good news and the bad news. If $\lambda_g > \lambda_b$ (so the good news arrive faster than the bad news), then "no news" leads the designer to form a pessimistic inference on the quality of the movie, with the posterior falling. By contrast, if $\lambda_g < \lambda_b$, then "no news" leads to on optimistic inference, with the posterior rising. We label the former case **good news** case and the latter **bad news** case. Recall that main body of the paper treats the special case of $\lambda_b = 0$, a pure good news case.

Let g_t and b_t denote the probability that the designer's belief is 1 and 0, respectively. Given the experimentation rate μ_t, these probabilities evolve according to

$$\dot{g}_t = (1 - g_t - b_t)\lambda_g \mu_t p_t, \tag{19}$$

with the initial value $g_0 = 0$, and

$$\dot{b}_t = (1 - g_t - b_t)\lambda_b \mu_t (1 - p_t), \tag{20}$$

with the initial value $b_0 = 0$.[1] Further, these beliefs must form a martingale:

$$p_0 = g_t \cdot 1 + b_t \cdot 0 + (1 - g_t - b_t)p_t. \tag{21}$$

The designer chooses the policy (α, β, γ), measurable, to maximize social welfare, namely

$$\mathcal{W}(\alpha, \beta, \chi) := \int_{t\geq 0} e^{-rt} g_t \gamma_t (1-c)\mathrm{d}t + \int_{t\geq 0} e^{-rt} b_t \beta_t (-c)\mathrm{d}t + \int_{t\geq 0} e^{-rt}(1 - g_t - b_t)\alpha_t (p_t - c)\mathrm{d}t,$$

where (p_t, g_t, b_t) must follow the required laws of motion: (18), (19), (20), and (21), where $\mu_t = \rho + \alpha_t$ is the total experimentation rate and r is the discount rate of the designer.[2]

Given policy (α, β, γ), conditional on being recommended to watch the movie, the agent will have the incentive to watch the movie, if and only if the expected quality of the movie—the posterior that it is good—is no less than the cost, or

$$\frac{g_t \gamma_t + (1 - g_t - b_t)\alpha_t p_t}{g_t \gamma_t + b_t \beta_t + (1 - g_t - b_t)\alpha_t} \geq c. \tag{22}$$

The following is immediate:

Lemma 1. *It is optimal for the designer to disclose the breakthrough (both good and bad)*

[1] These formulae are derived as follows. Suppose the probability that the designer has seen the good news by time t and the probability that she has seen the bad news by t are respectively g_t and b_t. Then, the probability of the good news arriving by time $t + \mathrm{d}t$ and the probability of the bad news arriving by time $t + \mathrm{d}t$ are, respectively, and to the first-order,

$$g_{t+\mathrm{d}t} = g_t + \lambda_g \mu_t p_t \mathrm{d}t(1 - g_t - b_t) \text{ and } b_{t+\mathrm{d}t} = b_t + \lambda_b \mu_t (1 - p_t)\mathrm{d}t(1 - g_t - b_t).$$

Dividing these equations by $\mathrm{d}t$ and taking the limit as $\mathrm{d}t \to 0$ yields (19) and (20).

[2] More precisely, the designer is allowed to randomize over the choice of policy (α, β, γ) (using a relaxed control, as such randomization is defined in optimal control). A corollary of our results is that there is no gain for him from doing so.

news immediately. That is, an optimal policy has $\gamma_t \equiv 1, \beta_t \equiv 0$.

Proof. If one raises γ_t and lowers β_t, it can only raise the value of objective $\mathcal{W}$ and relax (22) (and do not affect other constraints). □

Throughout, it is convenient to define $\Delta := \frac{\delta}{\lambda_g} = \frac{\lambda_g - \lambda_b}{\lambda_g}$. Then, using $\ell_t = \frac{p_t}{1-p_t}$, (18) can be restated as:

$$\dot{\ell}_t = -\ell_t \Delta \lambda_g \mu_t, \quad \ell_0 := \frac{p_0}{1-p_0}. \tag{23}$$

The two other state variables, namely the posteriors g_t and b_t on the designer's belief, are pinned down by ℓ_t (and thus by p_t) at least when $\lambda_g \neq \lambda_b$ (*i.e.*, when no news is not informationally neutral.) (We shall remark on the case of the neutrality case $\Delta = 0$.)

Lemma 2. *If* $\Delta \neq 0$, *then*

$$g_t = p_0 \left(1 - \left(\frac{\ell_t}{\ell_0}\right)^{\frac{1}{\Delta}}\right) \text{ and } b_t = (1-p_0)\left(1 - \left(\frac{\ell_t}{\ell_0}\right)^{\frac{1}{\Delta}-1}\right).$$

Proof. Let $\kappa_t := p_0/(p_0 - g_t)$. Note that $\kappa_0 = 1$. Then, it follows from (19) and (21) that

$$\dot{\kappa}_t = \lambda_g \kappa_t \mu_t, \quad \kappa_0 = 1. \tag{24}$$

Dividing both sides of (24) by the respective sides of (23), we get,

$$\frac{\dot{\kappa}_t}{\dot{\ell}_t} = -\frac{\kappa_t}{\ell_t \Delta},$$

or

$$\frac{\dot{\kappa}_t}{\kappa_t} = -\frac{1}{\Delta}\frac{\dot{\ell}_t}{\ell_t}.$$

It follows that, given the initial condition,

$$\kappa_t = \left(\frac{\ell_t}{\ell_0}\right)^{-\frac{1}{\Delta}}.$$

We can then unpack κ_t to recover g_t, and from this we can obtain b_t via (21). □

Next, substitute g_t and b_t into (22) to obtain:

$$\alpha_t \leq \bar{\alpha}(\ell_t) := \min\left\{1, \frac{\left(\frac{\ell_t}{\ell_0}\right)^{-\frac{1}{\Delta}} - 1}{k - \ell_t}\ell_t\right\}, \tag{25}$$

if the normalized cost $k := c/(1-c)$ exceeds ℓ_t and $\bar{\alpha}(\ell_t) := 1$ otherwise.

The next lemma will figure prominently in our characterization of the second-best policy later.

Lemma 3. *If $\ell_0 < k$ and $\Delta \neq 0$, then $\bar{\alpha}(\ell_t)$ is zero at $t = 0$, and increasing in t, strictly so whenever $\bar{\alpha}(\ell_t) \in [0, 1)$.*[3]

Proof. We shall focus on

$$\tilde{\alpha}(\ell) := \frac{\left(\frac{\ell}{\ell_0}\right)^{-\frac{1}{\Delta}} - 1}{k - \ell}\ell.$$

Recall $\bar{\alpha}(\ell) = \min\{1, \tilde{\alpha}(\ell)\}$. Since ℓ_t falls over t when $\Delta > 0$ and rises over t when $\Delta < 0$. It suffices to show that $\tilde{\alpha}(\cdot)$ is decreasing when $\Delta > 0$ and increasing when $\Delta < 0$.

We make several preliminary observations. First, $\tilde{\alpha}(\ell) \in [0, 1)$ if and only if

$$1 - (\ell/\ell_0)^{\frac{1}{\Delta}} \geq 0 \text{ and } k\ell^{\frac{1}{\Delta}-1}\ell_0^{-\frac{1}{\Delta}} > 1. \tag{26}$$

Second,

$$\tilde{\alpha}'(\ell) = \frac{(\ell_0/\ell)^{\frac{1}{\Delta}} h(\ell, k)}{\Delta(k-\ell)^2}, \tag{27}$$

where

$$h(\ell, k) := \ell - k(1 - \Delta) - k\Delta(\ell/\ell_0)^{\frac{1}{\Delta}}.$$

Third, (26) implies that

$$\frac{\mathrm{d}h(\ell, k)}{\mathrm{d}\ell} = 1 - k\ell^{\frac{1}{\Delta}-1}\ell_0^{-\frac{1}{\Delta}} < 0, \tag{28}$$

on any range of ℓ over which $\tilde{\alpha} \leq 1$. Note

$$h(0, k) = -k(1 - \Delta) = -k\frac{\lambda_b}{\lambda_g} \leq 0. \tag{29}$$

It follows from (28) and (29) that $h(\ell, k) < 0$ for any $\ell \in (0, k)$ and $\tilde{\alpha}(\ell) \in [0, 1)$. By (27), this last fact implies that $\tilde{\alpha}'(\ell) < 0$ if $\Delta > 0$ and $\tilde{\alpha}'(\ell) > 0$ if $\Delta < 0$, as was to be shown. $\square$

[3] The case $\Delta = 0$ is similar: the same conclusion holds but $\bar{\alpha}$ need to be defined separately.

Substituting the posteriors from Lemma 2 into the objective function and using $\mu_t = \rho + \alpha_t$, and with normalization of the objective function, the second-best program is restated as follows:

$$[SB] \qquad \sup_{\alpha} \int_{t \geq 0} e^{-rt} \ell_t^{\frac{1}{\Delta}} \left(\alpha_t \left(1 - \frac{k}{\ell_t} \right) - 1 \right) \mathrm{d}t$$

subject to

$$\dot{\ell}_t = -\Delta \lambda_g (\rho + \alpha_t) \ell_t, \tag{30}$$

proposition

$$0 \leq \alpha_t \leq \bar{\alpha}(\ell_t). \tag{31}$$

Obviously, the first-best program, labeled $[FB]$, is the same as $[SB]$, except that the upper bound for $\bar{\alpha}(\ell_t)$ is replaced by 1. We next characterize the optimal recommendation policy. The precise characterization depends on the sign of Δ, *i.e.*, whether the environment is that of predominantly good news or bad news.

Specifically, we focus on the "bad news" environment: $\Delta < 0$.[4] As before, we perform a change of variable to produce the following program for the designer: For problem $i = SB, FB$,

$$\sup_{u} \int_{\ell^0}^{\infty} e^{-rt(\ell)} \ell^{\frac{1}{\Delta}-1} \left(\left(1 - \frac{k}{\ell} \right) (1 - \rho u(\ell)) - u(\ell) \right) \mathrm{d}\ell,$$

s.t. $t(\ell^0) = 0$,

$$t'(\ell) = -\frac{u(\ell)}{\Delta \lambda_g \ell},$$

$$u(\ell) \in \mathcal{U}^i(\ell),$$

where as before $\mathcal{U}^{SB}(\ell) := [\frac{1}{\rho+\alpha(\ell)}, \frac{1}{\rho}]$ and $\mathcal{U}^{FB}(\ell) := [\frac{1}{\rho+1}, \frac{1}{\rho}]$. Again, a solution exists by the Filippov-Cesari theorem (Cesari, 1983).

Proposition 7. *The first-best policy (absent any news) prescribes no experimentation until the posterior p rises to p_b^{**}, and then full experimentation at the rate of $\alpha(p) = 1$ thereafter, for $p > p_b^{**}$, where*

$$p_b^{**} := c \left(1 - \frac{rv}{\rho + r(v + \frac{1}{\lambda_b})} \right).$$

The second-best policy implements the first-best if $p_0 \geq c$ or if $p_0 \leq \hat{p}_0$ for some $\hat{p}_0 < p_b^{**}$.

[4]The results for the "good news" case are remarkably similar to those of Proposition 1 in the paper. The results for good news and the neutral news case are available upon request from the authors.

If $p_0 \in (\hat{p}_0, c)$, then the second-best policy prescribes no experimentation until the posterior p rises to p_b^*, and then maximal experimentation at the rate of $\bar{\alpha}(\frac{p}{1-p})$ thereafter for any $p > p_b^*$, where $p_b^* > p_b^{**}$. In other words, the second-best policy triggers experimentation at a later date and at a lower rate than does the first-best.

Proof. As before, the necessary conditions for the second-best policy now state that there exists an absolutely continuous function $\nu : [0, \ell^0]$ such that, for all ℓ, either

$$\psi(\ell) := -\phi(\ell) = \Delta\lambda_g e^{-rt(\ell)} \ell^{\frac{1}{\Delta}} \left(\rho\left(1 - \frac{k}{\ell}\right) + 1\right) - \nu(\ell) = 0, \tag{32}$$

or else $u(\ell) = \frac{1}{\rho+\alpha(\ell)}$ if $\psi(\ell) > 0$ and $u(\ell) = \frac{1}{\rho}$ if $\psi(\ell) < 0$.

Furthermore, we must have

$$\nu'(\ell) = -\frac{\partial \mathcal{H}(t, u, \ell, \nu)}{\partial t} = re^{-rt(\ell)} \ell^{\frac{1}{\Delta}-1} \left(\left(1 - \frac{k}{\ell}\right)(1 - \rho u(\ell)) - u(\ell)\right) \ (\ell - \text{a.e.}). \tag{33}$$

Finally, transversality at $\ell = \infty$ ($t(\ell)$ is free) implies that $\lim_{\ell\to\infty} \nu(\ell) = 0$.

Since $\psi(\ell) = -\phi(\ell)$, we get from (32) that

$$\psi'(\ell) = -e^{-rt(\ell)} \ell^{\frac{1}{\Delta}-2} \left(r(\ell - k) + \rho\Delta\lambda_g k + \lambda_g(\rho(\ell - k) + \ell)\right).$$

Letting $\tilde{\ell} := \left(1 - \frac{\lambda_g(1+\rho\Delta)}{r+\lambda_g(1+\rho)}\right) k$, namely the solution to $\psi(\ell) = 0$. Then, ψ is maximized at $\tilde{\ell}$, and is strictly quasi-concave. Since $\lim_{\ell\to\infty} h(\ell) = 0$, this means that there must be a cutoff $\ell_b^* < \tilde{\ell}$ such that $\psi(\ell) < 0$ for $\ell < \ell_b^*$ and $\psi(\ell) > 0$ for $\ell > \ell_b^*$. Hence, the solution is bang-bang, with $u(\ell) = 1/\rho$ if $\ell < \ell_b^*$, and $u(\ell) = 1/(\rho + \alpha(\ell))$ if $\ell > \ell_b^*$.

The first-best policy has the same cutoff structure, except that the cutoff may be different from ℓ_b^*. Let ℓ_b^{**} denote the first-best cutoff.

First-best policy: We shall first consider the first best policy. In that case, for $\ell > \ell_b^{**}$,

$$t'(\ell) = -\frac{1}{\Delta\lambda_g(1+\rho)\ell}$$

gives

$$e^{-rt(\ell)} = C_2 \ell^{\frac{r}{(1+\rho)\Delta\lambda_g}},$$

for some non-zero constant C_2. Then

$$\nu'(\ell) = -\frac{rk}{1+\rho} C_2 \ell^{\frac{r}{(1+\rho)\Delta\lambda_g} + \frac{1}{\Delta} - 2}$$

and $\lim_{\ell\to\infty}\nu(\ell)=0$ give

$$\nu(\ell) = -\frac{rk\Delta\lambda_g}{r+(1+\rho)(1-\Delta)\lambda_g}C_2\ell^{\frac{r}{(1+\rho)\Delta\lambda_g}+\frac{1}{\Delta}-1}.$$

So we get, for $\ell > \ell_b^{**}$,

$$\psi(\ell) = -\Delta\lambda_g C_2 \ell^{\frac{r}{(1+\rho)\Delta\lambda_g}}\ell^{\frac{1}{\Delta}-1}\left(\ell(1+\rho)-k\rho\right) + \frac{rk\Delta\lambda_g}{r+(1+\rho)(1-\Delta)\lambda_g}C_2\ell^{\frac{r}{(1+\rho)\Delta\lambda_g}+\frac{1}{\Delta}-1}.$$

Setting $\psi(\ell_b^{**})=0$ gives

$$\frac{k}{\ell_b^{**}} = \frac{r+(1+\rho)(1-\Delta)\lambda_g}{r+\rho(1-\Delta)\lambda_g} = \frac{r+(1+\rho)\lambda_b}{r+\rho\lambda_b} = 1+\frac{\lambda_b}{r+\rho\lambda_b},$$

or

$$p_b^{**} = c\left(1-\frac{rv}{\rho+r(v+\frac{1}{(1-\Delta)\lambda_g})}\right) = c\left(1-\frac{rv}{\rho+r(v+\frac{1}{\lambda_b})}\right).$$

Second-best policy. We now characterize the second-best cutoff. There are two cases, depending upon whether $\alpha(\ell)=1$ is incentive-feasible at the threshold ℓ_b^{**} that characterizes the first-best policy. In other words, for the first-best to be implementable, we should have $\bar{\alpha}(\ell^{**})=1$, which requires

$$\ell_0 \geq k\left(\frac{r+\rho\lambda_b}{r+(1+\rho)\lambda_b}\right)^{1-\Delta} =: \hat{\ell}_0.$$

Observe that since $\Delta<0$, $\hat{\ell}_0<\ell^{**}$. If $\ell_0\leq\hat{\ell}_0$, then the designer begins with no experimentation and waits until the posterior belief improves sufficiently to reach ℓ^{**}, at which point the agents will be asked to experiment with full force, *i.e.*, with $\bar{\alpha}(\ell)=1$, that is, given that no news has arrived by that time. This first-best policy is implementable since, given the sufficiently favorable prior, the designer will have built sufficient "credibility" by that time. Hence, unlike the case of $\Delta>0$, the first best can be implementable even when $\ell_0<k$.

Suppose $\ell_0<\hat{\ell}_0$. Then, the first-best is not implementable. That is, $\bar{\alpha}(\ell_b^{**})<1$. Let ℓ_b^* denote the threshold at which the constrained designer switches to $\bar{\alpha}(\ell)$. We now prove that $\ell_b^*>\ell_b^{**}$.

For the sake of contradiction, suppose that $\ell_b^*\leq\ell_b^{**}$. Note that $\psi(x)=\lim_{\ell\to\infty}\phi(\ell)=0$. This means that

$$\int_{\ell_b^*}^{\infty}\psi'(\ell)\mathrm{d}\ell = \int_{\ell_b^*}^{\infty}e^{-rt(\ell)}\ell^{\frac{1}{\Delta}-2}\left((r+\lambda_b\rho)k-(r+\lambda_g(\rho+1))\ell\right)\mathrm{d}\ell = 0,$$

where $\psi'(\ell) = -\phi'(\ell)$ is derived using the formula in (32).

Let t^{**} denote the time at which ℓ_b^{**} is reached along the first-best path. Let

$$f(\ell) := \ell^{\frac{1}{\Delta}-2}\left((r+\lambda_b\rho)k - (r+\lambda_g(\rho+1))\ell\right).$$

We then have

$$\int_{\ell_b^*}^{\infty} e^{-rt^{**}(\ell)} f(\ell)\mathrm{d}\ell \geq 0, \tag{34}$$

(because $\ell_b^* \leq \ell_b^{**}$; note that $f(\ell) \leq 0$ if and only if $\ell > \tilde{\ell}$, so h must tend to 0 as $\ell \to \infty$ from above), yet

$$\int_{\ell_b^*}^{\infty} e^{-rt(\ell)} f(\ell)\mathrm{d}\ell = 0. \tag{35}$$

Multiplying $e^{rt^{**}(\tilde{\ell})}$ on both sides of (34) gives

$$\int_{\ell_b^*} e^{-r(t^{**}(\ell)-t^{**}(\tilde{\ell}))} f(\ell)\mathrm{d}\ell \geq 0. \tag{36}$$

Likewise, multiplying $e^{rt(\tilde{\ell})}$ on both sides of (35) gives

$$\int_{\ell_b^*}^{\infty} e^{-r(t(\ell)-t(\tilde{\ell}))} f(\ell)\mathrm{d}\ell = 0. \tag{37}$$

Subtracting (36) from (37) gives

$$\int_{\ell_b^*} \left(e^{-r(t(\ell)-t(\tilde{\ell}))} - e^{-r(t^{**}(\ell)-t^{**}(\tilde{\ell}))}\right) f(\ell)\mathrm{d}\ell \leq 0. \tag{38}$$

Note $t'(\ell) \geq (t^{**})'(\ell) > 0$ for all ℓ, with strict inequality for a positive measure of ℓ. This means that $e^{-r(t(\ell)-t(\tilde{\ell}))} \leq e^{-r(t^{**}(\ell)-t^{**}(\tilde{\ell}))}$ if $\ell > \tilde{\ell}$, and $e^{-r(t(\ell)-t(\tilde{\ell}))} \geq e^{-r(t^{**}(\ell)-t^{**}(\tilde{\ell}))}$ if $\ell < \tilde{\ell}$, again with strict inequality for a positive measure of ℓ for $\ell \geq \ell_b^{**}$ (due to the fact that the first best is not implementable; *i.e.*, $\bar{\alpha}(\ell_b^{**}) < 1$). Since $f(\ell) < 0$ if $\ell > \tilde{\ell}$ and $f(\ell) > 0$ if $\ell < \tilde{\ell}$, we have a contradiction to (38).

The sufficiency can be proven by using Arrow sufficiency theorem (Seierstad and Sydsæter, 1987, Theorem 5, p.107). The detail is omitted. □

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