

Multidimensional matching*

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Abstract

We present general mathematical methods used in the study of multidimensional matching problems with transferable utility, paying particular attention to the case in which the dimensions of heterogeneity on the two sides of the market are unequal. We then describe an approach to solve a new subclass of these problems: problems where agents on one side of the market are multidimensional and agents on the other side are uni-dimensional. Lastly, we analyze several examples, including an hedonic model with differentiated products, a marriage market model where wives are differentiated in income and fertility, and a competitive variation of the Rochet-Choné problem. In the latter example, we show that the bunching phenomena, observed by Rochet and Choné in the monopoly context, do not occur in the competitive context.

1 Introduction

Matching problems under transferable utility have attracted considerable attention in recent years within economic theory. The general goal

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is to understand the stable equilibria of matches between distributions of agents on two sides of a ‘market’ (for example, husbands and wives in the marriage market, CEOs and firms in the labor market, producers and consumers in a market for differentiated commodities, etc.), as well as the resulting division of surplus between partners. Until recently, most work has focused on the setting in which a single characteristic is used to distinguish between the agents on each side; for example, in the marriage market, several models assume that individuals differ only by their income (or their human capital). These models have the advantage of being analytically tractable, and often allow explicit closed form solutions. Under the classical Spence-Mirrlees condition, the only stable matching is the positive assortative one, the nature of which (‘who marries whom’) is directly determined by the underlying distributions of the male and female characteristic. However, these one dimensional models are unsatisfactory in many situations, as both casual empiricism and factual evidence indicate that agents often match on several traits. In the marriage market, for instance, the suitability of a potential marriage between a woman and man typically depends on several characteristics of both, including income and education, but also age, tastes, ethnic background, physical attractiveness, etc.

It is therefore important to study and understand *multidimensional* matching problems, in which agents on both side of the market are differentiated using several characteristics. These models have garnered increasing visibility in recent years, due to their wider applicability and flexibility, but their introduction brings forth serious theoretical challenges. The nature of the equilibrium matching is more interesting but also more complex; in contrast to the one dimensional case, it is no longer determined by the sole knowledge of the distributions of individual characteristics, even under (a generalization of) the Spence-Mirrlees condition. From a more technical perspective, it is generally not possible to derive closed form solutions; and discretising matching problems leads to a linear program, which often become numerically unwieldy when type spaces are multidimensional.

The purpose of this paper is to provide a general characterization of multidimensional matching models, in terms of existence, uniqueness and qualitative properties of stable matches. Since the work of [Shapley & Shubik (1971)] in the discrete setting, and [Gretsky, Ostroy & Zame (1992)] in the continuum, it has been understood that transferable utility matching is equivalent to a variational problem; this problem is known in the mathematics literature as the *Monge-Kantorovich optimal transport* problem. Our main claim is that the geometrical and topological structure induced on the underlying spaces by the surplus function can pro-

vide powerful insights into matching patterns. Specifically, the considerable existing literature on multidimensional optimal transport¹ can be exploited to derive conditions under which stable matches are unique and pure, and to understand their local geometry.

We put a particular emphasis on the case in which the dimensions of heterogeneity on the two sides of the market are unequal (say, $m > n$). These sorts of problems have received relatively little attention from the mathematics community, but are quite natural economically; the dimension essentially reflects the number of attributes used to distinguish between agents and there is no compelling reason in general to expect this number to coincide for agents on the two different sides of the market (say, for consumers and producers). A typical pattern emerges in these situations, since for one side of the market (the one with a lower dimension), identical agents are typically matched with a continuum of different partners. We explore the topology of the ‘indifference sets’ thus defined, and provide conditions under which they can be expected to be smooth manifolds of dimension $m - n$.

Of specific interest are the so-called ‘multi-to-one dimensional matching problems’, in which agents on one side of the market are assumed to be multidimensional, while those on the other side are unidimensional. They include an economically important class of examples (see, for instance, [Chiappori, Oreffice & Quintana-Domeque (2012)] and [Low (2014)]), for which one can obtain explicitly the stable matchings. In this context, we describe a general approach aimed at characterizing the equilibrium matching. We show that the problem can be expressed in terms of a system of partial differential equations,² and provide a robust methodology which allows, under suitable conditions, to explicitly characterize its solutions. We discuss some interesting features that the indifference sets exhibit, and which are typically absent in purely unidimensional problems. For instance, the optimal mapping may be discontinuous, and so women of similar types may marry men of very different types.

We then consider three specific examples that illustrate different potential applications of matching models. One is a standard, hedonic model of the type used, in particular, in the empirical IO literature. While these models typically assume imperfect competition, we show how a competitive version can be analyzed. Under standard conditions, existence, but also uniqueness and purity, obtain naturally. In particular, we show that the mathematical notion of purity has in this context a natural interpretation in terms of ‘bunching’. In addition, it is in general possible to derive closed form solutions for the pricing schedule; we

¹For a general survey, see for instance [Villani (2009)] and [Santambrogio (2015)].

²Or a single partial differential equation in case $m = 2 = n + 1$.

illustrate how this can be done using specific distributions.

Our second example considers a context where women are characterized by two traits (socio-economic status, from now on SES, and fertility) whereas men only differ by their SES - a case recently studied by [Low (2014)]. Then it may be the case that small changes in a middle class woman's SES (while her fertility remains constant) result in a large change in her husband's income. Again, we provide a general characterization of the solution and describe some of its qualitative features.

Lastly, the multi-to-one dimensional framework naturally leads to investigating the relationship between matching models and principal-agents problems under multidimensional asymmetric information. While the general problem remains widely open (and quite challenging), we illustrate our contribution by considering a competitive variant of the seminal model of [Rochet & Choné (1998)], in which goods can be produced by a 1-dimensional, heterogeneous distribution of producers, rather than a single monopolist. Unlike the original Rochet-Choné framework, this model is equivalent to a matching problem of our form, and can be solved explicitly by the method described above. We provide a full characterization of the resulting equilibrium price schedule. In particular, we show that in our competitive framework, and in contrast to the original monopolist setting, there is never bunching: consumers of different types *always* buy goods with different characteristics. In other words, the strange bunching patterns emphasized by Rochet and Choné do not appear to be intrinsically linked to the multidimensional nature of the adverse selection problem; rather, they are due to the distortions created by the presence of a monopolist producer.

2 Multidimensional matching under transferable utility: basic properties

2.1 General framework and basic results

2.1.1 The model

We consider sets $X \subseteq \mathbf{R}^m$ and $Y \subseteq \mathbf{R}^n$, parametrizing populations of agents on two sides of a market. In what follows, we shall stick to the marriage market interpretation (so that X and Y will denote the set of potential wives and husbands respectively), although alternative interpretations are obviously possible. They are distributed according to probability measures μ on X and ν on Y , respectively. In the transferable utility framework, a potential matching of agents $x \in X$ and $y \in Y$ generates a combined surplus $s(x, y)$, where $s : X \times Y \rightarrow \mathbf{R}$.

This surplus can be divided in any way between the agents x and y . For simplicity, we assume that s and its derivatives are smooth and bounded unless otherwise remarked; many of the results we describe can also be extended to surpluses with less smoothness, as in [Chiappori, McCann & Nesheim (2010)] and [Noldeke & Samuelson (2015)] for example.

A *matching* is characterized by a probability measure γ on the product $X \times Y$, whose marginals are μ and ν , that is

$$\gamma(A \times Y) = \mu(A) \quad \text{and} \quad \gamma(X \times B) = \nu(B) \quad (1)$$

for all Borel $A \subset X, B \subset Y$. Intuitively, a matching is an assignment of the agents in the sets X and Y into pairs, and $\gamma(x, y)$ is related to the probability that x will be matched to y ; in particular, $\gamma(x, y) = 0$ implies that agents x and y are not matched together. The marginal condition is often called the *market clearing* criterion. We denote the set of all matchings by $\Gamma(\mu, \nu)$.

Integrable functions $u : X \rightarrow \mathbf{R}$ and $v : Y \rightarrow \mathbf{R}$ are called payoff functions corresponding to γ if they satisfy the *budget constraint*:

$$u(x) + v(y) \leq s(x, y) \quad (2)$$

γ almost everywhere — i.e., for any pair of agents who match with positive probability. For such a pair $(x, y) \in \text{spt } \gamma$,³ the functions $u(x)$ and $v(y)$ are interpreted respectively as the indirect utilities derived from the match by agents x and y ; the constraint (2) ensures that the total indirect utility $u(x) + v(y)$ collected by the two agents does not exceed the total surplus $s(x, y)$ available to them.

A matching γ is called *stable* if there exist payoff functions $u(x)$ and $v(y)$ satisfying both (2) and the reverse inequality

$$u(x) + v(y) - s(x, y) \geq 0 \quad (3)$$

for all $(x, y) \in X \times Y$. Condition (3) expresses the stability of the matching in the following sense; if we had $u(x) + v(y) < s(x, y)$ for any (currently unmatched) pair of agents, it would be desirable for each of them to leave their current partners and match together, dividing the excess surplus $s(x, y) - u(x) - v(y) > 0$ in such a way as to increase the payoffs to both x and y . Note that (2) and (3) together ensure $u(x) + v(y) = s(x, y)$, γ almost everywhere: if two agents match with positive probability, then they split the surplus generated between them.

³Here $\text{spt } \gamma$ refers to the support of γ , i.e. the smallest closed set containing the full mass of γ .

For simplicity, we shall assume complete participation is incentivized throughout and that supply balances demand (reflected by the fact that μ and ν have equal mass); when these assumptions are violated it is well-known that they can be restored by augmenting both sides of the market with a fictitious type representing the outside option of remaining unmatched (see for instance [Chiappori, McCann & Nesheim (2010)]).

Given a stable match γ and associated matching functions u, v , the set

$$S = \{(x, y) \in X \times Y \mid u(x) + v(y) = s(x, y)\}$$

is of particular interest; as $\text{spt } \gamma \subset S$, it tells us which agents can match together. If S is concentrated on a graph $\{(x, F(x)) \mid x \in S\}$ of some function $F : X \rightarrow Y$, the stable matching is called *pure*, the interpretation being that almost all agents of type x must match with agents of the same type $y = F(x)$; in particular, purity excludes the presence of *randomization*, whereby an agent may be randomly assigned to different partners. In this case, the distribution ν agrees with the image $F_{\#}\mu$ of μ under F , which assigns mass

$$(F_{\#}\mu)(V) := \mu[F^{-1}(V)] \tag{4}$$

to each $V \subset Y$.⁴ More generally, it is interesting and useful to understand the geometry of S .

Although u and v will not generally be everywhere differentiable, some mild regularity condition guarantees differentiability almost everywhere, as stated by the following result:

Theorem 1 *If the surplus function s is Lipschitz, so are the payoffs u and v — and with the same Lipschitz constant; if $s \in C^2(X \times Y)$, then u and v have second-order Taylor expansions, Lebesgue almost-everywhere.*

Proof. See [Villani (2009)] or [Santambrogio (2015)]. ■

When the probability measures μ and ν come from Lebesgue densities, this almost-everywhere differentiability proves sufficient for many analytic purposes. We use $\text{Dom } Du$ (respectively $\text{Dom } D^2u$) to denote those $x \in X$ at which u has a first- (respectively second-)order Taylor expansion, and $\text{Dom}_0 D^i u := (\overline{X})^0 \cap \text{Dom } D^i u$ where \overline{X} and X^0 denote the closure and interior of X , respectively.

The fact that S is the zero-set of the non-negative function (3) enters crucially. It implies in particular the first-order and second order conditions

$$(Du(x), Dv(y)) = (D_x s(x, y), D_y s(x, y)) \tag{5}$$

⁴Also called the *push-forward* $F_{\#}\mu$ of μ through F ; see e.g. [Ahmad, Kim & McCann (2011)].

and

$$\begin{pmatrix} D^2u(x) & 0 \\ 0 & D^2v(y) \end{pmatrix} \geq \begin{pmatrix} D_{xx}^2s(x, y) & D_{xy}^2s(x, y) \\ D_{yx}^2s(x, y) & D_{yy}^2s(x, y) \end{pmatrix} \quad (6)$$

are satisfied at each $(x, y) \in S \cap (X \times Y)^0$ for which the derivatives in question exist; here X^0 denotes the interior of X , and inequality (6) should be understood to mean the difference of these $(m+n) \times (m+n)$ symmetric matrices is non-negative definite.

The first-order condition

$$Du(x) = D_x s(x, y) \quad (7)$$

(for example) has an interesting, economic interpretation. In a transferability utility model, the wife's share $u(x)$ of the surplus comes at the expense of her husband y 's share. Thus (7) expresses the equality of his marginal willingness $Du(x)$ to pay for variations in her qualities $x = (x^1, \dots, x^m)$ with the couple's marginal surplus $D_x s(x, y)$ for the same variations. Similarly

$$Dv(y) = D_y s(x, y) \quad (8)$$

equates her marginal willingness to pay for variations in his qualities $y = (y^1, \dots, y^n)$ with their marginal surplus for such variations. This has important consequences for situations where characteristics are not exogenously given but result from some investment made by individuals before the beginning of the game (human capital being an obvious example). Then (7) implies that the marginal return, for the individual, of an investment in characteristics is exactly equal to the social return (defined as the contribution of the investment to aggregate surplus). In other words, such investments are typically efficient, despite being made non-cooperatively before the matching game; their impact on global welfare is internalized by matching mechanisms, a point made by [Cole, Mailath & Postlewaite (2001)] and [Iyigun & Walsh (2007)] among others.

2.1.2 Variational interpretation: optimal transport and duality

The problem of identifying stable matches turns out to have a variational formulation, known as the optimal transport, or Monge-Kantorovich, problem in the mathematics literature (see for instance [Villani (2009)] and [Santambrogio (2015)]). This is the problem of matching the measures μ and ν so as to maximize the total surplus; that is, to find γ among the set $\Gamma(\mu, \nu)$ which maximizes

$$s[\gamma] := \int_{X \times Y} s(x, y) d\gamma(x, y). \quad (\text{MK})$$

The following theorem can be traced back to [Shapley & Shubik (1971)] for finite type spaces X and Y , and to [Gretsky, Ostroy & Zame (1992)] more generally. It asserts an equivalence between (MK) and stable matchings.

Theorem 2 *A matching measure $\gamma \in \Gamma(\mu, \nu)$ is stable if and only if it maximizes (MK).*

As the maximization of a linear functional over a convex set, problem (MK) has a dual problem, which is useful both in studying its maximizers, and in clarifying its relation with stable matching. The dual problem to (MK) is to minimize

$$\mu[u] + \nu[v] := \int_X u(x) d\mu(x) + \int_Y v(y) d\nu(y). \quad (\text{MK}_*)$$

among functions $u \in L^1(\mu)$ and $v \in L^1(\nu)$ satisfying the stability condition (3). It is well known that under mild conditions, duality holds (see, for instance, [Villani (2009)]), that is:

$$\max_{\gamma \text{ satisfying (1)}} s[\gamma] = \min_{(u,v) \text{ satisfying (3)}} (\mu[u] + \nu[v]). \quad (9)$$

Note that for any u and v satisfying the stability constraint (3) and any matching $\gamma \in \Gamma(\mu, \nu)$, the marginal condition implies

$$\mu[u] + \nu[v] = \int_{X \times Y} (u(x) + v(y)) d\gamma(x, y) \geq \int_{X \times Y} s(x, y) d\gamma(x, y)$$

and we can have equality if and only if $u(x) + v(y) = s(x, y)$ holds γ -almost everywhere. It then follows from the duality theorem that γ is a maximizer in (MK) (and hence a stable match) and u, v are minimizers in the dual problem (MK_{*}), precisely when $u(x) + v(y) = s(x, y)$ holds γ almost everywhere; in other words, the solutions to (MK_{*}) coincide with the payoff functions.

An immediate corollary is the following:

Corollary 3 *Let s and \bar{s} be two surplus functions. Assume there exists two functions f and g , mapping \mathbf{R}^m to \mathbf{R} and \mathbf{R}^n to \mathbf{R} respectively, such that*

$$s(x, y) = \bar{s}(x, y) + f(x) + g(y)$$

For any measures μ and ν , any stable matching for s is a stable matching for \bar{s} and conversely.

Proof. Any stable measure γ for s solves the surplus maximization problem:

$$\max_{\gamma \text{ satisfying (1)}} \int_{X \times Y} s(x, y) d\gamma(x, y). \quad (10)$$

which is equivalent to:

$$\max_{\gamma \text{ satisfying (1)}} \int_{X \times Y} \bar{s}(x, y) d\gamma(x, y) + \int_X f(x) d\mu(x) + \int_Y g(y) d\nu(y)$$

The last two integrals are given (by the marginal conditions on γ), and so any γ that solves (10) also solves (11):

$$\max_{\gamma \text{ satisfying (1)}} \int_{X \times Y} \bar{s}(x, y) d\gamma(x, y). \quad (11)$$

■

An important consequence of this result is that *the observation of matching patterns can only (at best) identify the surplus up to two additive functions of x and y respectively*. We shall see later on that, in general, s cannot be identified even up to two such additive functions.

Problem (MK) has been studied extensively over the past 25 years, and Theorem 2 allows the application of a large resulting body of theory to the stable matching problem. In particular, conditions on s leading to existence, uniqueness and purity of the solution to (MK) are well known and surveyed in [Villani (2009)]. These properties can be compactly and elegantly expressed in terms of the *cross difference*, introduced by [McCann (2014)] and defined on $(X \times Y)^2$ by:

$$\delta(x, y, x_0, y_0) = s(x, y) + s(x_0, y_0) - s(x, y_0) - s(x_0, y)$$

The relevance of the cross difference to problem (MK) will become more apparent in what follows. For now, we hint at its role by noting that in one dimension, $m = n = 1$, positivity of the cross difference whenever $(x - x_0) \cdot (y - y_0) > 0$ is equivalent to supermodularity of s (the Spence-Mirrlees condition). For more general X and Y , the cross difference is zero along the diagonal $\{(x, y) = (x_0, y_0)\}$ and nonnegative along $\text{spt } \gamma \times \text{spt } \gamma \subseteq (X \times Y)^2$ for any stable match γ . This non-negativity of s on $(\text{spt } \gamma)^2$ is typically referred to as *s -monotonicity* of $\text{spt } \gamma$, and is a special case of a more general condition called *s -cyclical monotonicity*, which characterizes the support of optimal matchings.

2.1.3 Existence, purity and uniqueness of a stable matching

The variational formulation is quite helpful in establishing the basic properties of stable matchings. Existence, for instance, can now be asserted using basic continuity and compactness arguments, as stated by the following result:

Theorem 4 Assume $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ are bounded and $s \in C(X \times Y)$. Then there exists an optimizer γ to (MK), and therefore a stable match.

Proof. See [Villani (2009)] or [Santambrogio (2015)] ■

We now consider uniqueness and purity. Aside from its theoretical interest, uniqueness of the optimal matching γ plays an important computational role, as in its absence more sophisticated techniques must be employed. In practice, solutions are often assumed to be pure in empirical studies. Since this conclusion is not generically satisfied [McCann & Rifford (preprint)], it is desirable to know conditions on s , μ and ν which guarantee it. Furthermore, in hedonic contexts, purity implies the absence of 'bunching' (whereby different agents consume the same product); this property will be particularly important when we investigate the interactions between matching and contract theory.

Note, first, that uniqueness is not guaranteed in general. For instance, when the surplus function takes the additive form $s(x, y) = f(x) + g(y)$, the functional

$$s[\gamma] = \int_{X \times Y} s(x, y) d\gamma(x, y) = \int_X f(x) d\mu(x) + \int_Y g(y) d\nu(y)$$

is *constant* throughout the set $\Gamma(\mu, \nu)$ of feasible matching measures, and so any matching γ is optimal and hence stable. It is therefore clear that certain structural conditions on s are indeed needed to ensure purity and uniqueness.

The key condition for purity of the optimal matching is a nonlocal generalization of the Spence-Mirrlees condition, known as the *twist* condition:

Definition 5 The function $s \in C^1$ satisfies the twist condition if, for each fixed $x_0 \in X$ and $y_0 \neq y \in Y$, the mapping

$$x \in X \mapsto \delta(x, y, x_0, y_0) \tag{12}$$

has no critical points.

By differentiating δ with respect to x and rearranging, we see that this is equivalent to

$$D_x s(x, y) \neq D_x s(x, y_0) \tag{13}$$

for all x and distinct $y \neq y_0$. The twist condition is therefore equivalent to the *injectivity* of $y \mapsto D_x s(x, y)$, for each fixed x . This injectivity

in turn implies that the husband type y of woman type $x \in \text{Dom } Du$ is uniquely determined by his marginal willingness $Du(x)$ to pay for her qualities through the first-order condition (7). For instance, in a one-dimensional context ($m = 1 = n$), the classical Spence-Mirrlees condition imposes that either $\frac{\partial^2 s}{\partial x \partial y} > 0$ or $\frac{\partial^2 s}{\partial x \partial y} < 0$ over $X \times Y$, which implies that $y \mapsto \frac{\partial s}{\partial x}(x, y)$ is strictly monotone (and hence injective) for each fixed x . It is in this sense that the twist condition can be viewed as a non-local generalization of the Spence-Mirrlees condition.

The twist condition is sufficient to guarantee purity, as stated by the following result:

Theorem 6 ([Gangbo (1995)], [Levin (1999)]) *Assume that μ is absolutely continuous with respect to Lebesgue measure and the surplus s satisfies the twist condition. Then any solution γ to (MK) is pure.*

Proof. Let γ solve (MK) for a twisted surplus $s \in C^1$. According to Theorems 1 and 2, there exist Lipschitz potentials u on X and v on Y satisfying $u(x) + v(y) \geq s(x, y)$ for all $(x, y) \in X \times Y$, with equality holding γ -a.e. Being Lipschitz, u is differentiable Lebesgue almost everywhere, hence μ -a.e., where μ denotes the left marginal of γ . Thus for μ -a.e. x we find $Du(x) = D_x s(x, y)$ as in (5). The twist condition (12) implies we can invert this relation to write $y = F(x)$, where $F(x) = [D_x s(x, \cdot)]^{-1}(Du(x))$. This shows γ is pure. ■

The absolute continuity of μ is a technical condition required to ensure the utilities can be differentiated on a set of full μ measure; the payoff functions are guaranteed to be Lipschitz if the surplus is, and are hence differentiable Lebesgue almost everywhere by Rademacher's theorem (but not everywhere in general). The condition on the measure can be weakened somewhat, but some regularity is needed: as a simple counterexample, if $\mu = \delta_{x_0}$ is a Dirac mass but ν is not, then the optimal matching (indeed, the only measure in $\Gamma(\mu, \nu)$) is product measure $\delta_{x_0} \otimes \nu$, which pairs every point y with x_0 and is certainly not pure.

Three remarks can be made at this point. First, if s is twice continuously differentiable and Y has nonempty interior, the twist condition immediately implies that $n \leq m$, as it asserts the existence of a smooth injection (13) from an open subset of \mathbf{R}^n into \mathbf{R}^m . Second, it is worth noting that in many relevant situations, the twist condition does indeed fail; for example, if we replace X and Y with compact smooth manifolds, it fails for *any* smooth surplus function s . Third, the twist condition is *not* necessary for purity. For instance, [Kitagawa & Warren (2012)] provide a setting in which purity holds in the absence of twist.

One can readily see that purity implies uniqueness:

Corollary 7 *Under the conclusions of the preceding theorem, the optimal matching γ is unique.*

Proof. Suppose two solutions γ_0 and γ_1 to (MK) exist. Convexity of the problem makes it clear that $\gamma_2 = (\gamma_0 + \gamma_1)/2$ is again a solution. The conclusion asserts that γ_2 concentrates on the graph of a map $F : X \rightarrow Y$, and vanishes outside this graph. Non-negativity ensures the same must be true for γ_0 and γ_1 . But then $\gamma_0 = (F \times id)_{\#}\mu = \gamma_1$ by Lemma 3.1 of [Ahmad, Kim & McCann (2011)] ■

The converse to this Corollary is not true; i.e., one can easily find situations where the optimal matching is unique but not pure. Additional conditions which ensure uniqueness, but not purity, of the optimal matching can be found in [Chiappori, McCann & Nesheim (2010)] and [McCann & Rifford (preprint)].

2.1.4 An example

As an illustration of Theorem 6, we consider a particular case that has been widely used in empirical applications ([Galichon & Salanié (2012)], [Galichon & Dupuy (2014)], [Lindenlaub (2015)] to name just a few). Assume that $m = n$, and that the surplus takes the form:

$$s(x, y) = f_X(x) + g_Y(y) + \sum_{k=1}^n f_k(x_k) g_k(y_k)$$

Two remarks can be made about this form. First, we may, without loss of generality, disregard the first two terms by assuming that $f_X = g_Y = 0$; the stable measure will not be affected. Second, this form is necessary and sufficient for the matrix of cross derivatives:

$$D_{xy}^2 s = \left(\left(\frac{\partial^2 s}{\partial x_k \partial y_l} \right) \right)$$

to be diagonal (a case investigated by [Lindenlaub (2015)]).

In that case, for any $y \neq \bar{y}$, we have that:

$$D_x s(x, y) - D_x s(x, \bar{y}) = \begin{pmatrix} f'_1(x_1)(g_1(y_1) - g_1(\bar{y}_1)) \\ \vdots \\ f'_n(x_n)(g_n(y_n) - g_n(\bar{y}_n)) \end{pmatrix}$$

In particular (and assuming that none of the f_i 's have anywhere vanishing derivatives), the twist condition is satisfied whenever the g_i 's are strictly monotonic. We conclude that, in that case, the stable match is unique and pure; that is, there exists a function F mapping \mathbf{R}^m to itself, such that x is matched a.s. with $y = (F_1(x), \dots, F_m(x))$. In addition,

[Lindenlaub (2015)] show that if the f and g are strictly increasing, then F_i is strictly increasing in x_i for all i (a property she calls multidimensional positive assortative matching).

2.1.5 Local structure of the optimal matching

In the absence of the twist condition, or in the case where both marginals are singular, one may not expect purity of the optimal match. However, under a generic nondegeneracy criterion, which is a local generalization of the Spence-Mirrlees condition, something can be still be asserted about the local geometric structure of its support. For a fixed (x_0, y_0) , let H be the Hessian of the function $(x, y) \mapsto \delta(x, y, x_0, y_0)$, evaluated at the point $(x, y) = (x_0, y_0)$. A simple calculation yields that H takes the block form:

$$H = \begin{bmatrix} 0 & D_{xy}^2 s \\ D_{yx}^2 s & 0 \end{bmatrix}$$

where

$$D_{xy}^2 s = \left(\frac{\partial^2 s}{\partial x_i \partial y_j} \right)_{ij}$$

is the $m \times n$ matrix of mixed second order partials, and the 0's in the upper left and lower right hand corners represent $m \times m$ and $n \times n$ vanishing blocks, respectively. Denoting by r the rank of $D_{xy}^2 s$, we have the following theorem (see [McCann, Pass & Warren (2012)] and [Pass (2012a)]).

Theorem 8 *There exists a neighborhood N of (x_0, y_0) such that the intersection $N \cap S$ is contained in a Lipschitz submanifold of $X \times Y$ of dimension $m + n - r$. At points where S is differentiable it is spacelike in the sense that $H(v, v) \geq 0$ for any v which is tangent to the set S .*

In particular, in the special case where $m = n$ and H has full rank, $r = n$, the support is at most $m = n$ dimensional (locally). When these dimensions differ, and the $n \times m$ matrix has full rank $r = \min(n, m)$, the dimension is $\max(n, m)$ and its codimension is $\min(n, m)$ [Pass (2012a)]. If $r = \min(n, m)$ we say s is *non-degenerate* at (x_0, y_0) .

The spacelike assertion of Theorem 8 can be useful to identify the orientation of the optimal matching, as we will see below.

Example 9 *When $m = n = 1$, assuming the Spence-Mirrlees condition $\frac{\partial^2 s}{\partial x \partial y} > 0$, the theorem tells us that the solution concentrates on a 1-dimensional Lipschitz curve $(x(t), y(t))$. Wherever that curve is differentiable, the spacelike condition boils down to $x'(t) \frac{\partial^2 s}{\partial x \partial y} y'(t) \geq 0$, or $x'(t)y'(t) \geq 0$, yielding positive assortative matching. There is only one matching γ with this structure, and it can be computed explicitly from*

the measures μ and ν (provided μ has no atoms); it is given by $F_{\#}\mu$ where $F : X \rightarrow Y$ satisfies

$$\int_{(-\infty, F(x))} d\nu \leq \int_{-\infty}^x d\mu \leq \int_{(-\infty, F(x)]} d\nu$$

In words: x is matched with $y = F(x)$ such that the number of women whose characteristic is larger than x equals the number of men above y .

2.1.6 Regularity (smoothness) of the matching function

When s satisfies the twist condition, and the optimal matching is therefore pure and unique, it is interesting to ask whether the mapping $F : X \rightarrow Y$ generating the matching is continuous; qualitatively, this is the question of whether women x and x' whose types are ‘close’ must marry men of similar characteristics.

This turns out to generally not be the case, as a wide range of examples throughout the literature on optimal transport show. In particular, as was shown by [Ma, Trudinger & Wang (2005)], it is necessary (but not sufficient) that for each $x \in X$, the set $D_x s(x, Y) = \{D_x s(x, y) \mid y \in Y\} \subset \mathbf{R}^n$ be convex; if this condition fails, it is possible to construct measures μ and ν with smooth positive densities for which F is discontinuous.

When $m > n$, this convexity is particularly restrictive; it can be shown to fail, *unless* the function s takes a reducible, or *index* form, $s(x, y) = b(I(x), y)$, for some $I : X \rightarrow \mathbf{R}^n$ [Pass (2012b)]. We will illustrate this phenomenon with an example later on.

Even when this convexity holds, the optimal map may not be continuous in general; in this case, their regularity is governed by a restrictive fourth order differential condition on s , known as the Ma-Trudinger-Wang condition (see [Ma, Trudinger & Wang (2005)] and [Loeper (2009)]). [Kim & McCann (2010)] show this condition to be equivalent to a sign condition on the curvature of the space $X \times Y$ geometrized using the Hessian H as a (pseudo)-Riemannian metric.⁵

2.2 Extensions

2.2.1 Individual utilities

The stability condition allows information on individual utilities at the stable match to be recovered. To see why, note first that stability implies that, for μ almost every x ,

$$u(x) = \max_y (s(x, y) - v(y)). \quad (14)$$

⁵See [McCann (2014)] for a general overview of the regularity of optimal mappings.

Assume, now, that the matching is pure (say, because the twist condition is satisfied). The envelope theorem then yields, wherever u is differentiable and $y = F(x)$ is matched with x ,

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial s}{\partial x_i}(x, F(x)). \quad (15)$$

which gives the partials of u , and therefore defines u up to an additive constant. Note, incidentally, that these partial differential equations must be compatible, which generates restrictions on the matching function F ; namely, assuming double differentiability:

$$\sum_k \frac{\partial^2 s}{\partial x_i \partial y_k} \frac{\partial F_k}{\partial x_j} = \sum_k \frac{\partial^2 s}{\partial x_j \partial y_k} \frac{\partial F_k}{\partial x_i} \quad (16)$$

$$\geq \frac{\partial^2 s}{\partial x^i \partial x^j}(x, F(x)) \quad (17)$$

where $y_k = F_k(x)$ and the partials of s are taken at $(x, F(x))$ and the inequality is from (6). In particular, in the case of multi to one dimensional matching, then $y = F(x_1, \dots, x_m)$, and (16) becomes a system of partial differential equations that F must satisfy (which reduces to a single equation in case $m = 2 = n + 1$); together with the measure restrictions and the matrix inequality (17), this typically identifies the matching function F . We shall come back to this point in Section 4.

2.2.2 Index and pseudo-index models

A very special case, which has been largely used in practical applications, is obtained when the surplus function s is weakly separable in one vector of characteristics. Assume, indeed, that there exist two functions I and σ , mapping \mathbf{R}^n to \mathbf{R} and \mathbf{R}^{m+1} to \mathbf{R} respectively, such that:

$$s(y, x) = \sigma(x, I(y)). \quad (18)$$

In words, the various male characteristics y affect the matching function only through some one dimensional index $I(y)$. This assumption is quite restrictive; in practice, it requires that men with different characteristics y and y' but the same index (i.e., $I(y) = I(y')$) be perfect substitutes on the matching market *for any potential spouse* x .

Formally, if s is smooth, then the index form requires that s satisfies the following conditions:

$$\frac{\partial}{\partial x_m} \left(\frac{\partial s / \partial y_k}{\partial s / \partial y_l} \right) = 0 \quad \forall k, l, m.$$

These conditions express the fact that the marginal rate of substitution between y_k and y_l (which defines the slope of tangent to the corresponding iso-surplus curve) does not depend on x ; indeed, (18) implies that:

$$\frac{\partial s / \partial y_k}{\partial s / \partial y_l} = \frac{\partial I / \partial y_k}{\partial I / \partial y_l}.$$

The main practical interest of index models is that, whenever (18) is satisfied, the matching problem is de facto one-dimensional in y ; technically, one can replace the space Y and the measure ν with $\tilde{Y} = \text{Im}I \subset \mathbb{R}$ and the *push-forward* $\tilde{\nu} := I_{\#}\nu$ of ν through I defined as in (4). In particular, when the index property (18) is satisfied, then the matching problem boils down to a multi-to-one dimensional problem, of the type discussed in Section 4.⁶

Another notable property of index models, that was observed in [Pass (2012b)], is that when $m \leq n$, the convexity of the sets $\{D_y s(y, x) \mid x \in X\}$, necessary to ensure that the matching function is continuous, *requires* that the function s takes the form $s(x, y) = \sigma(x, I(y))$, where here $I : \mathbb{R}^n \rightarrow \mathbb{R}^m$. In particular, when $m = 1$, s must have an index form. Therefore, when $n > m = 1$, if a surplus function s is not of index form, there are absolutely continuous measures μ and ν (with smooth densities) for which the matching function F is discontinuous; we will see an example of this below.

Lastly, the notion of index model can for some purposes be slightly relaxed. Specifically, we define a *pseudo-index* model by assuming that there exist three functions α , I and σ , mapping \mathbf{R}^n to \mathbf{R} , \mathbf{R}^n to \mathbf{R} and \mathbf{R}^{m+1} to \mathbf{R} respectively, such that:

$$s(x, y) = \alpha(y) + \sigma(x, I(y)). \quad (19)$$

Here, male characteristics y affect the matching function through *two* one-dimensional indices $\alpha(y)$ and $I(y)$. The crucial remark, however, is the following. Assume that $D_x \sigma(x, i)$ is injective in i ; this simply requires that $\partial \sigma / \partial x_k(x, i)$ is strictly monotonic in i for at least one k . Then:

$$D_x s(x, y) = D_x \sigma(x, I(y)) \neq D_x \sigma(x, I(y_0)) = D_x s(x, y_0)$$

for any y, y_0 such that $I(y) \neq I(y_0)$. It follows from the proof of the twist theorem that the support of the stable measure is born by the graph of

⁶A practical difficulty is that, for most empirical applications, the index I is not known ex ante and has to be empirically estimated. See [Chiappori, Oreffice & Quintana-Domeque (2012)] for a precise discussion.

a function. Although the stable matching on $\mathbb{R}^m \times \mathbb{R}^n$ is not pure (since all males with the same index are perfect substitutes), for the ‘reduced’ matching problem defined on $\mathbb{R}^m \times \mathbb{R}$ by the surplus function $\sigma(x, i)$ and the measures μ and $I_{\#}\nu$, the stable matching is pure - i.e., there exists a function ϕ such that any woman x is matched with probability one with a man whose index is $i = \phi(x)$. In particular, most results derived in the multi to one dimensional case (see Section 4 below) still apply in that case.

2.2.3 Links with hedonic models

Next, we briefly recall the canonical link between matching and hedonic models,⁷ which will be crucial for some of our applications (particularly the competitive IO model and the competitive version of Rochet-Choné). An hedonic model involves three sets: a set X of buyers (endowed with a measure μ), a set Y of sellers (endowed with a measure ν) and a set Z of products; intuitively, a product is defined by a finite vector of characteristics, and buyers purchase one product (at most) in any period - think of a car or a house, for instance. Both buyers and sellers are price-takers, and consider the price $P(z)$ of any product $z \in Z$ as given. A buyer with a vector of characteristics x maximizes a quasi-linear utility of the form

$$U = u(x, z) - P(z)$$

while producer y maximizes profit

$$\Pi = P(z) - c(y, z)$$

where $c(y, z)$ is the cost, for producer y to produce product z .

Remark 10 *It is important to note that the producer’s profit depends on his characteristics, on the product’s characteristics and on the price, but not on the characteristics of the buyer. This ‘private value’ aspect will be crucial for the relationship with bidimensional adverse selection that we discuss below.*

An hedonic equilibrium is defined as a measure α on the product set $X \times Y \times Z$, whose first and second marginals are μ and ν , and a function P such that, for any $(\bar{x}, \bar{y}, \bar{z}) \in \text{spt } \alpha$, then

$$\bar{z} \in \arg \max_z (u(x, z) - P(z)) \cap \arg \max_z (P(z) - c(y, z)).$$

⁷See [Chiappori, McCann & Nesheim (2010)], [Ekeland (2005)] and [Ekeland (2010)].

In words, α represents an assignment of buyers and sellers to each other and to products, and an equilibrium is reached if the product assigned to x (resp. y) maximizes x 's utility (y 's profit).

To see the link with matching models, define the pairwise surplus function

$$s(x, y) = \sup_{z \in Z} u(x, z) - c(y, z). \quad (20)$$

and consider the matching model defined by (X, Y, s) . Then [Chiappori, McCann & Nesheim (2010)] prove the following results:

- for any equilibrium of the hedonic model, the projection of the measure α on the set $X \times Y$, together with the functions

$$U(x) = \max_z u(x, z) - P(z) \quad \text{and} \quad V(y) = \max_z (P(z) - c(y, z))$$

form a stable matching.

- Conversely, for any γ that solves the matching problem, there exist a price function P that satisfies

$$\inf_{y \in Y} \{v(y, z) + r(y)\} \geq P(z) \geq \sup_{x \in X} \{u(x, z) - q(x)\} \quad (21)$$

With $\alpha \equiv (id_X \times id_Y \times z_0)_\# \gamma$, where $z_0 = z_0(x, y) \in \arg \max_z u(x, z) - c(y, z)$, any such P forms an equilibrium pair (α, P) .

2.2.4 Testability

Lastly, let's briefly consider the important issue of testability: what testable restrictions (if any) are generated by the general matching structure described above? Obviously, the answer depends on what we can observe. Consider the simplest case, in which we only observe matching patterns ('who marries whom'). Technically, we are now facing an inverse problem: knowing the spaces X and Y and the measure γ , can we find a surplus s for which γ is stable? This question should however be slightly rephrased to rule out degenerate solutions; for instance, any measure is stable for the degenerate surplus $s(x, y) = 0 \quad \forall x, y$.

We therefore consider the following problem: Given two spaces X , Y and some measure γ on $X \times Y$, is it always possible to find a surplus s such that γ is the *unique* stable matching of the matching problem (X, Y, s) ?

A first remark is that if we impose enough 'regularity' in the model, the answer is positive. Specifically, let us consider the case in which the support of the measure is born by the graph of some function F , and that F is non degenerate (in the sense that the derivative of F has full

rank over the entire space). Then one can always find a surplus for which γ is the unique stable matching: we just need to take $s(x, y) = -|F(x) - y|^2/2$. Indeed, γ obviously maximizes the primal, optimal transportation problem, which guarantees stability; moreover, the surplus satisfies the twist condition, which guarantees uniqueness. The corresponding payoffs are $u(x) = 0 = v(y)$.

However, non degeneracy is crucial for this result to hold. For one thing, if F is degenerate, the twist condition does not hold, and while γ is always stable for $s(x, y) = -|F(x) - y|^2/2$, it may not be the unique stable matching. Moreover, and while Theorem 8 implies that any stable matching for a non-degenerate surplus concentrates on a set of dimension at most $\max(m, n)$, it is possible to find measures supported on sets of this dimension which are *not* stable for any C^2 , *non-degenerate* surplus. To see this, consider the $m = n = 1$ case; let $X = Y = (0, 1) \subseteq \mathbf{R}$. Non-degeneracy here simply means $\frac{\partial^2 s}{\partial x \partial y} \neq 0$, which implies either $\frac{\partial^2 s}{\partial x \partial y} > 0$ everywhere (so s is super-modular) or $\frac{\partial^2 s}{\partial x \partial y} < 0$ everywhere (so s is sub-modular). In these two cases, it is well known that stable matches concentrate on monotone increasing or decreasing sets, respectively. Therefore, any γ concentrating on a set of dimension $\max(m, n) = 1$ (for instance, a smooth curve), which is neither globally increasing nor decreasing (for example, the curve $(y = 4(x - 1/2)^2)$), cannot be stable for any non-degenerate surplus.

3 Matching with unequal dimensions

We now turn special attention to the case in which the dimensions $m \geq n$ of heterogeneity on the two sides of the market are unequal. In this case, one expects many-to-one rather than one-to-one matching. In a companion publication, we develop a detailed mathematical theory for this situation, focusing especially on the case $m > n = 1$. Here we announce only the main conclusions of that theory and a few of the underlying ideas, suppressing technical details wherever possible to be able to move quickly on to interpretation and applications.

When $m \geq n$, it is natural to expect that at equilibrium the subset $F^{-1}(y) \subset X \subset \mathbf{R}^m$ of partners which a man of type $y \in \text{Dom}_0 Dv$ is indifferent to will generically have dimension $m - n$, or equivalently, codimension n . We are interested in specifying conditions under which this indifference set will in fact be a smooth submanifold.⁸ Let us explore this situation, paying particular attention to the separate roles played by the surplus function $s(x, y)$ as opposed to the populations μ and ν in

⁸In the familiar case $n = m$ it would then consist of one (or more) isolated points — a single one if s happens to be twisted.

determining these indifference sets.

3.1 Potential Indifference Sets

For any equilibrium matching γ and payoffs (u, v) , we have already seen that $(x, y) \in S \cap (X \times \text{Dom}_0 Dv)$ implies (8). That is, all partner types $x \in X$ for husband $y \in \text{Dom}_0 Dv$ lie in the same level set of the map $x \mapsto D_y s(x, y)$. If we know $Dv(y)$, we can determine this level set precisely; it depends on μ and ν as well as s . However, in the absence of this knowledge it is useful to define the *potential indifference sets*, which for given $y \in Y$ are merely the level sets of the map $x \in X \mapsto D_y s(x, y)$. We can parameterize these level sets by (cotangent) vectors $k \in \mathbf{R}^n$:

$$X(y, k) := \{x \in X \mid D_y s(x, y) = k\}, \quad (22)$$

or we can think of $y \in Y$ as inducing an equivalence relation between points of X , under which x and $\bar{x} \in X$ are equivalent if and only if

$$D_y s(x, y) = D_y s(\bar{x}, y).$$

Under this equivalence relation, the equivalent classes take the form (22). We call these equivalence classes *potential indifference sets*, since they represent a set of partner types which $y \in \text{Dom } Dv$ has the potential to be indifferent between. The equivalence class containing a given partner type $\bar{x} \in X$ will also be denoted by

$$L_{\bar{x}}(y) = X(y, D_y s(\bar{x}, y)) = \{x \in X \mid D_y s(x, y) = D_y s(\bar{x}, y)\}. \quad (23)$$

A key observation concerning potential indifference sets is the following proposition.

Definition 11 (Surplus degeneracy) *Given $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$, we say $s \in C^2(X \times Y)$ degenerates at $(x, y) \in X \times Y$ if $\text{rank}(D_{xy}^2 s(\bar{x}, \bar{y})) < \min\{m, n\}$. Otherwise we say s is non-degenerate at (\bar{x}, \bar{y}) .*

Proposition 12 (Structure of potential indifference sets) *Let $s \in C^{r+1}(X \times Y)$ for some $r \geq 1$, where $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ with $m \geq n$. If s does not degenerate at $(\bar{x}, \bar{y}) \in X \times Y$, then \bar{x} admits a neighbourhood $U \subset \mathbf{R}^m$ such that $L_{\bar{x}}(\bar{y}) \cap U$ coincides with the intersection of X with a C^r -smooth, codimension n submanifold of U .*

Proof. Since $s \in C^2$, the surplus extends to a neighbourhood $U \times V$ of (\bar{x}, \bar{y}) on which s continues to be non-degenerate (by lower semi-continuity of the rank). The set $\{x \in U \mid D_y s(x, y) = D_y s(\bar{x}, \bar{y})\}$ forms a codimension n submanifold of U , by the preimage theorem

[Guillemin & Pollack (1974), §1.4]. More specifically, the rank condition implies that choosing a suitable orthonormal basis for \mathbf{R}^m yields $\det[\frac{\partial^2 s}{\partial x^i \partial y^j}(\bar{x}, \bar{y})]_{1 \leq i, j \leq n} \neq 0$. In these coordinates, the potential indifference set is locally parameterized as the inverse image under the C^r map $x \in U \mapsto (D_y s(x, \bar{y}), x_{n+1}, \dots, x_m)$ of the affine subspace $\{D_y s(\bar{x}, \bar{y})\} \times \mathbf{R}^{n-m}$. Taking U and V smaller if necessary, the inverse function theorem then shows $L_{\bar{x}}(\bar{y}) \cap U$ to be C^r . ■

Although we have stated the proposition in local form, it implies that if $\bar{k} = D_y s(\bar{x}, \bar{y})$ is a *regular value* of $x \in X \mapsto D_y s(x, \bar{y})$ — meaning $D_{xy}^2 s(x, \bar{y})$ has rank n throughout $L_{\bar{x}}(\bar{y})$ — then $L_{\bar{x}}(\bar{y}) = X(\bar{y}, \bar{k})$ is the intersection of X with an $m - n$ dimensional submanifold of \mathbf{R}^m . Note however that this proposition says nothing about points (\bar{x}, \bar{y}) where s degenerates, which can happen throughout $\text{spt } \gamma$.

3.2 Potential versus actual indifference sets

As argued above, the potential indifference sets (22) and (23) are determined by the surplus function $s(x, y)$ without reference to the populations μ and ν to be matched. On the other hand, the indifference set actually realized by each $y \in Y$ depends on the relationship between μ , ν and s . This dependency is generally complicated, as illustrated by the following example.

Example 13 *Consider the surplus function:*

$$s(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_1 y_2$$

where $X \subset \mathbf{R}^3, Y \subset \mathbf{R}^2$. The potential indifference sets are given, for any $k \in \mathbf{R}^2$, by:

$$X(y, k) := \left\{ x \in X \mid \begin{array}{l} x_1 + x_3 y_2 = k_1 \\ \text{and} \\ x_2 + x_3 y_1 = k_2 \end{array} \right\}. \quad (24)$$

These are straight lines in \mathbf{R}^3 , parallel to the vector $\begin{pmatrix} y_2 \\ y_1 \\ -1 \end{pmatrix}$. Therefore,

for any given $y \in \mathbf{R}^2$, we know that the set of spouses matched with y (the indifference set corresponding to husband y) will be such a straight line. However, it is certainly not true that any such line (obtained for an arbitrary choice of k) will be an indifference set curve. For a given y , the exact equation of the indifference set corresponding to y is defined by the value of the specific vector k which is relevant for that particular y — and this depends on the measures μ and ν .

However, there is one case in the problem may simplify substantially: the case of multi-to-one dimensional matching, namely $n = 1$. In this case, suppose $D_{xy}^2 s(\cdot, y)$ is non-vanishing (i.e. $\frac{\partial s}{\partial y}(\cdot, y)$ takes only regular values). Then the potential indifference sets $X(y, k)$ are codimension 1 in \mathbf{R}^m ; that is, they are curves in \mathbf{R}^2 , surfaces in \mathbf{R}^3 , and hypersurfaces in higher dimensions $m \geq 4$. Moreover, as k moves through \mathbf{R} , these potential indifference sets sweep out more and more of the mass of μ . For each $y \in Y$ there will be some choice of $k \in \mathbf{R}$ for which the μ measure of $\{x \mid D_y s(x, y) \leq k\}$ exactly coincides with the ν measure of $(-\infty, y]$ (assuming both measures are absolutely continuous with respect to Lebesgue, or at least that μ concentrates no mass on hypersurfaces and ν has no atoms). In this case the potential indifference set $X(y, k)$ is said to split the population proportionately at y , making it a natural candidate for being the true indifference set $F^{-1}(y)$ to be matched with y .⁹ In the next Section, we go on to describe and contrast situations in which this expectation is born out and leads to a complete solution from those in which it does not.

4 Multi-to-one dimensional matching

We now explain a new approach to a specific class of models, largely unexplored in either the mathematics or economics literature, but which can often be solved explicitly with the techniques outlined below and developed more fully in [Chiappori, McCann & Pass (preprint)]. These are *multi-to-one dimensional* models, in which agents on one side of the market (say wives) are bi-dimensional (or, potentially, higher dimensional) while agents on the other side (husbands) are one-dimensional. Thus, we are matching a distribution on $x = (x_1, \dots, x_m) \in \mathbf{R}^m$ with another on $y \in \mathbf{R}$. The surplus s is then a function $s(x_1, \dots, x_m, y)$ of $m + 1$ real variables, and will typically be increasing in each argument.

The crucial notion, in this setting, is that of a *iso-husband curve*, defined as the indifference set of a given husband y , i.e. as the submanifold of wives among which husband y turns out to be indifferent facing the given market conditions. Iso-husband curves, as we shall see, play a key role in the construction of an explicit solution to the matching/optimal transportation problem. In addition, a crucial property of these curves is that they can in principle be empirically identified; see [Chiappori, Orefice & Quintana-Domeque (2012)] for a detailed discussion. In fact, it has been argued that the theoretical properties of iso-husband curves could provide the most powerful empirical tests of match-

⁹Since $k = s_y(x, y)$ can be recovered from any $x \in X(y, k)$ and y , we may equivalently say x splits the population proportionately at y , and vice versa.

ing theory (see for instance [Chiappori, Oreffice & Quintana-Domeque (preprint)]).

The goal is to construct from data (s, μ, ν) a matching function $F : X \rightarrow Y \subset \mathbf{R}$, whose level sets $F^{-1}(y)$ constitute iso-husband curves. At the end of the preceding section we identified a natural candidate for this indifference set: namely the potential indifference set which divides the mass of μ in the same ratio as y divides ν ; whether or not these natural candidates actually fit together to form the level sets of a function or not depends on a subtle interaction between μ , ν and s . When they do, we say the model is *nested*, and in that case we show that the resulting function $F : X \rightarrow Y$ produces a stable equilibrium match.

Mathematically, the simplest multi-to-one dimensional models arise from index surpluses, which have economic motivation arising from [Chiappori, Oreffice & Quintana-Domeque (preprint)]. These models are nested for *every* choice of the distributions ν and μ , as we demonstrate below. Moreover, in the following section, we will discuss three applications of multidimensional matching theory; the first two of these deal explicitly with multi-to-one dimensional problems. The first model arises in the marriage market, where recent research indicates that it is appropriate to model women using both education and fertility and men using income only [Low (2014)][Low (preprint)]. A second example comes from a hedonic variant of the Rochet-Choné screening problem.

4.1 Constructing explicit solutions for nested data

We now give a heuristic description of a general algorithm for constructing the solution to the matching problem when one of the dimensions is $n = 1$; the aim is to find a solution to (16)-(17) above (or, equivalently, (15)), which satisfies the mass balance condition $\nu = F_{\#}\mu$ and the space-like condition in Theorem 8. In order to work, our approach requires the nestedness property mentioned above and detailed below. These conditions are satisfied in a wide class of multi-to-one dimensional matching problems; they are illustrated in the theorem and examples presented below. However, except in the Spence-Mirrlees (with $m = n = 1$) and in the index and pseudo-index cases, this nestedness depends not only on s , but also on μ and ν .

For each fixed $y \in Y \subseteq \mathbf{R}$, our goal is to identify the iso-husband (or indifference) set $\{x \in X \mid F(x) = y\}$ of husband type y facing the given market conditions. When differentiability of v holds at y , the argument in the preceding section implies that this is contained in one of the potential indifference sets $X(y, k)$ from (22). Proposition 12 indicates when this set will have codimension 1; it generally divides X into two

pieces: the sublevel set

$$X_{\leq}(y, k) := \{x \in X \mid \frac{\partial s}{\partial y}(x, y) \leq k\}, \quad (25)$$

and its complement $X_{>}(y, k) := X \setminus X_{\leq}(y, k)$. We denote its strict variant by $X_{<}(y, k) := X_{\leq}(y, k) \setminus X(y, k)$.

To select the appropriate level set, assuming F is differentiable, the spacelike condition implies $\frac{\partial F}{\partial x_i}(x) \frac{\partial^2 s}{\partial x_i \partial y} \geq 0$ for each $i = 1, \dots, m$. Summing on i suggests points \bar{x} in the region $X_{\leq}(y, k)$ get paired with points $\bar{y} \leq y$. We therefore choose the unique level set *splitting the population proportionately* with y ; that is, the $k = k(y)$ for which the μ measure of female types $X_{\leq}(y, k)$ coincides with the ν measure of male types $(-\infty, y]$. We then set $y := F(x)$ for each x in the set $X(y, k)$.

Our first theorem specifies conditions under which the resulting match $\gamma = (id \times F)_{\#}\mu$ optimizes the Kantorovich problem (MK); we view it as the natural generalization of the positive assortative matching results of [Mirrlees (1971)] [Becker (1973)] and [Spence (1973)] from the one-dimensional to the multi-to-one dimensional setting. Unlike their criterion, which depends only on s , ours relates s to μ and ν , by requiring the sublevel sets $y \in Y \mapsto X_{\leq}(y, k(y))$ identified by the procedure above to depend monotonically on $y \in \mathbf{R}$, with the strict inclusion $X_{\leq}(y, k(y)) \subset X_{\leq}(y', k(y'))$ holding whenever $\nu[(y, y')] > 0$. We say the model (s, μ, ν) is *nested* in this case. Our situation is naturally more complicated than theirs, since there is no obvious ordering of the women's types, but generally a variety of possible orderings depending on population frequencies μ and ν ; nestedness rather asserts that the women's preferences enjoy some degree of compatibility, in the sense that for $[y, \bar{y}] \subset Y$ any wife $\bar{x} \in X(\bar{y}, k(\bar{y}))$ assigned to the higher type of husband has a greater willingness to pay for variations in the qualities of husband type \bar{y} (and similarly of husband type y) than does any wife $\underline{x} \in X(y, k(y))$ assigned to the lower type of husband.

Theorem 14 (Optimality of nested matchings) *Let $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}$ be connected open sets equipped with Borel probability measures μ and ν . Assume ν has no atoms and μ vanishes on each C^1 hypersurface. Use $s \in C^2(X \times Y)$ and $s_y = \frac{\partial s}{\partial y}$ to define X_{\leq} , $X_{<}$ etc., as in (25)*

Assume s is non-degenerate, $|D_x s_y| \neq 0$, throughout $X \times Y$. Then for each $y \in \bar{Y}$ there is a maximal interval $K(y) = [k^-(y), k^+(y)] \neq \emptyset$ such that $\mu[X_{\leq}(y, k)] = \nu[(-\infty, y)]$ for all $k \in K(y)$. Both k^+ and $-k^-$ are upper semicontinuous. Assume both maps $y \in Y \mapsto X_{\leq}(y, k^{\pm}(y))$ are non-decreasing, and moreover that $\int_y^{y'} d\nu > 0$ implies $X_{\leq}(y, k^+(y)) \subseteq X_{<}(y', k^-(y'))$. Then $k^+ = k^-$ holds ν -a.e. Setting $F(x) = y$ for each

$x \in X(y, k^+(y))$ defines a stable match $F : X \rightarrow Y$ [μ -a.e.]. Moreover, $\gamma = (id \times F)_{\#}\mu$ maximizes (MK) uniquely on $\Gamma(\mu, \nu)$. Finally, if $\text{spt } \nu$ is connected then F extends continuously to X .

Idea of proof. Non-degeneracy implies $X(y, k) := X_{\leq}(y, k) \setminus X_{<}(y, k)$ is an $m - 1$ dimensional C^1 submanifold of X orthogonal to $D_x s_y(x, y) \neq 0$. Since both μ and ν vanish on hypersurfaces, the function

$$h(y, k) := \mu[X_{\leq}(y, k)] - \nu[(-\infty, y)] \quad (26)$$

is continuous, and for each $y \in Y$ climbs monotonically from $-\nu[(-\infty, y)]$ to $1 - \nu[(-\infty, y)]$ with $k \in \mathbf{R}$. This proves the existence of $k^{\pm}(y)$ and confirms the zero set of $h(y, k)$ is closed. Thus k^- is lower semicontinuous, k^+ is upper semicontinuous, and by the intermediate value theorem $[k^-(y), k^+(y)]$ is non-empty.

The main strategy for the rest of the proof is to use $k^+(y)$ to construct a Lipschitz equilibrium payoff function v by solving $v^{'+}(y)$ a.e. Together with u from (14), it can be shown that (u, v) minimizes the dual problem (MK $_{*}$) and $\gamma = (id \times F)_{\#}\mu$ maximizes the planners problem (MK). For details and the case $s \in C^{1,1}$, see [Chiappori, McCann & Pass (preprint)].

■

Note that nestedness is a property of the three-tuple (s, μ, ν) ; that is, for most surplus functions, the model may or may not be nested depending on the measures under consideration. In some cases, however, the surplus function is such that the model is nested for all measures (s, μ, ν) . This is the case for the *pseudo-index models* defined above. Indeed, assume that the surplus has the form:

$$s(x_1, x_2, y) = \alpha(x_1, x_2) + \sigma(I(x_1, x_2), y). \quad (27)$$

Then

$$\frac{\partial s}{\partial y}(x, y) = \frac{\partial \sigma}{\partial y}(I(x_1, x_2), y)$$

only depends on (x_1, x_2) through the one-dimensional index $I(x)$. It follows that the iso-husband sets are defined by equations of the type $I(x) = k(y)$, which do not depend on y . Such curves cannot intersect as soon as $k(y)$ is strictly monotonic in y .^{10,11}

¹⁰In the un-nested case, it may still be possible to solve the problem using ad hoc methods. For example, in some cases it can be determined apriori that γ must couple certain subsets of X with certain subsets of Y . Then, the method above may be applied successfully to these subsets, even if it fails when applied to the whole of X and Y . We demonstrate this with an example (see Remark 5.1.4 below).

¹¹A similar scheme was developed in [Pass (2012b)], and under strong conditions on the marginals μ and ν and the surplus s , it was proven that this yields the optimal solution.

In [Chiappori, McCann & Pass (preprint)] we show the converse is also true: a non-degenerate surplus is pseudo-index if and only if (s, μ, ν) is nested for all choices of absolutely continuous population densities μ and ν .

4.2 Criteria for nestedness

The preceding theorem illustrates the powerful implications of nestedness, when it is present. For suitable data, the next theorem and corollaries give characterizations of nestedness which are often simpler to check in practice. These are based on a description of the motion of the iso-husband set in response to changes in the husband type, which is obtained using the theory of level set dynamics. Once again, sharper statements and detailed proofs may be found in [Chiappori, McCann & Pass (preprint)]. There the conclusion of following theorem is also shown to be sharp, in the sense that dk/dy diverges at the endpoints of Y whenever the area of the iso-husband sets $F^{-1}(y)$ shrinks to zero. Since we intend to apply the theorem on Lipschitz domains, we first recall the definition of transversality in that context.

Definition 15 (Transversality) *Recall $\bar{X}(y, k(y))$ intersects $\partial X \in C^1$ non-transversally if their normals coincide (or equivalently, are multiples of each other) at some point of intersection. If ∂X is merely Lipschitz, they intersect non-transversally if the inward or outward normal to $\bar{X}(y, k(y))$ is a generalized normal to ∂X , meaning it can be expressed as limit of convex combinations of outward normals at arbitrarily close points where ∂X is differentiable.*

Theorem 16 (Dependence of iso-husbands on husband type) *Let $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}$ be connected Lipschitz domains, equipped with Borel probability measures $d\mu(x) = f(x)dx$ and $d\nu(y) = g(y)dy$ whose Lebesgue densities satisfy $\log f \in C^1(X)$ and $\log g \in C_{loc}^0(Y)$. Assume $s \in C^3$ and non-degenerate throughout $\bar{X} \times Y$. Then the functions k^\pm of Theorem 14 coincide. Moreover $k := k^\pm \in C_{loc}^1(Y \setminus Z)$ outside the relatively closed set*

$$Z := \{y \in Y \mid \bar{X}(y, k(y)) \text{ intersects } \partial X \text{ non-transversally}\}. \quad (28)$$

As $y \in Y \setminus Z$ increases the outward normal velocity of $X_\leq(y, k(y))$ at $x \in X(y, k(y))$ is given by $(k' - s_{yy})/|D_x s_y|$.

Sketch of proof. Under the additional regularity assumed in the theorem, we differentiate (26) at $y \notin Z$ to obtain

$$h_k := \frac{\partial h}{\partial k} = \int_{X(y, k)} f(x) \frac{d\mathcal{H}^{m-1}(x)}{|D_x s_y(x, y)|} > 0 \quad \text{and}$$

$$h_y := \frac{\partial h}{\partial y} = -g(y) - \int_{X(y,k)} f(x) \frac{s_{yy}(x,y)}{|D_x s_y(x,y)|} d\mathcal{H}^{m-1}(x), \quad (29)$$

where \mathcal{H}^{m-1} is Hausdorff $m - 1$ dimensional (surface) measure. The formula for h_k is a straightforward application of the co-area formula from [Evans & Garipey (1992)]; it can also be seen as a consequence of the fundamental theorem of calculus, after applying the implicit function theorem to $s_y(x, y) = k$ to get $|D_x s_y|^{-1}$ as the outward normal velocity at x of $X_{\leq}(y, k)$ when k is increased as y is held fixed. The corresponding formula (29) for h_y results from the fact that the analogous Lipschitz velocity is given by $-s_{yy}/|D_x s_y|$ when y is increased as k is held fixed. Note that because $y \notin Z$ the limits from the left and the right which define the derivatives (29) agree; we need not include any additional contributions to these integrals coming from $\bar{X}(y, k) \cap \partial X$ satisfying $s_y(x, y) = k$. The continuous dependence of h_k and h_y on (y, k) are shown in [Chiappori, McCann & Pass (preprint)], after which $k \in C_{loc}^1$ can be inferred by applying the implicit function theorem to $h(y, k(y)) = 0$. ■

As consequences of this description, we obtain two alternate characterizations of nestedness in [Chiappori, McCann & Pass (preprint)]:

Corollary 17 (Dynamic criteria for nestedness) *Under the hypotheses of Theorem 16: if the model is nested then $k' - s_{yy} \geq 0$ for all $y \in Y \setminus Z$ and $x \in X(y, k(y))$, with strict inequality holding at some x for each y . Conversely, if $Z = \emptyset$ and strict inequality holds for all $y \in Y$ and $x \in X(y, k(y))$, then the model is nested.*

Corollary 18 (Unique splitting criterion for nestedness) *A model (s, μ, ν) satisfying the hypotheses of Theorem 16 with $Z = \emptyset$ is nested if and only if each $x \in X$ corresponds to a unique $y \in Y$ splitting the population proportionately, i.e. which satisfies*

$$\int_{X_{\leq}(y, s_y(x, y))} d\mu = \int_{-\infty}^y d\nu. \quad (30)$$

In this case, the stable matching is given by $F(x) = y$.

The first corollary states that the model is nested if and only if all iso-husband sets move outward as y is increased. Besides proving useful in the examples below, the second corollary shows nestedness is methodologically essential: without it, F fails to be well-defined (unless proportionate population splitting were to be augmented with some further criteria).

4.3 Smoothness of Payoffs and Matchings

Finally, we are able to address the questions of smoothness of the matching function $F : X \rightarrow Y$ in the nested case. In general dimension, this is a notoriously subtle and challenging question [Villani (2009)]. For $n = m > 1$ a fairly satisfactory regularity theory has been developed following work of [Ma, Trudinger & Wang (2005)]. But for $m > n = 1$ little is known, outside of the pseudo-index case [Pass (2012b)].

Assuming transversality ($Z = \emptyset$), Theorems 14 and 16 give conditions under which F is continuous and $k = dv/dk \in C_{loc}^1$ on the interiors of their respective domains. Recalling $v'(F(x)) = s_y(x, F(x))$ from (5), differentiation yields

$$(k'(F(x)) - s_{yy}(x, F(x)))DF(x) = D_x s_y(x, F(x)). \quad (31)$$

Thus we immediately see we can bootstrap continuous differentiability of the matching function $F \in C^1$ from continuity $F \in C^0$, wherever the normal velocity $k' - s_{yy}$ of the iso-husband set of x is *strictly* positive. Even assuming $s \in C^\infty$, to get more smoothness for F from this identity, we need more smoothness for k , or equivalently v .

Conditions guaranteeing $v \in C_{loc}^{r,1}(Y^0)$ (i.e. r times continuously differentiable, with Lipschitz derivatives) for any integer $r \geq 1$ are provided in [Chiappori, McCann & Pass (preprint)]. The overall strategy there is to use an induction on r to extract additional smoothness of the function h from (26), starting with that provided by Theorems 14 and 16. Since $h(y, k(y)) = 0$, this smoothness is transferred to $k = dv/dy$ using the implicit function theorem. To differentiate expressions like (29), we first use a suitably general form of the divergence theorem to rewrite them as

$$\begin{aligned} h_k &= \int_{X_{\leq}(y,k)} \nabla_X \cdot V d^m x - \int_{\overline{X_{\leq}(y,k)} \cap \partial X} V \cdot \hat{n}_X d\mathcal{H}^{m-1} \\ h_y &= -g(y) - \int_{X_{\leq}(y,k)} \nabla_X \cdot (s_{yy} V) d^m x + \int_{\overline{X_{\leq}(y,k)} \cap \partial X} s_{yy} V \cdot \hat{n}_X d\mathcal{H}^{m-1} \end{aligned}$$

where $V(x) = f s_{yy} \frac{D_x s_y}{|D_x s_y|^2} \Big|_{y=F(x)}$. Their derivatives are then given as integrals along the moving interfaces, weighted by normal velocities as in Theorem 16 and its proof. These in turn must be controlled by appropriate assumptions and a delicately structured inductive hypothesis.

4.4 Surplus identification in the nested case

Assume that we can observe iso-husband sets in a multiple market setting; what does it tell us about the surplus? We now give a precise

answer to that question. As already noted, if we only observe matching patterns, then the surplus s can be identified at best up to an additive function of x and an additive function of y . This, however, is not the only flexibility we have in defining s . To see why, remember that an arbitrary iso-husband set, with an equation of the form $y = F(x)$, lies in a level set

$$\frac{\partial s(x, F(x))}{\partial y} = k$$

for some constant $k = v'(y)$. Knowing the map F for a single pair (μ, ν) tells the direction (but not the magnitude) of $D_x s_y$ along the graph of F : assuming enough smoothness it is given by (31). To get the magnitude we need to know the husband's marginal share v'' of the payoff also.

Although maps F will not generally exist for all choices of pairs (μ, ν) (counterexamples are given in [Chiappori, McCann & Nesheim (2010)]), if we know the map F corresponding to enough choices of (μ, ν) , then by a suitable choice we can make the graph of F pass through any point (x', y') that we choose. Lemma 19 below shows that this can be done with sufficient smoothness for (31) to hold, and that it costs no generality to take μ uniform on a small ball around x . In this way, we learn the direction and sign (but not the magnitude, unless we also know the marginal payoffs v') of $D_x s_y$ globally.

If we also know the payoffs we can integrate $D_x s_y$ to find $s(x, y)$, up to an arbitrary additive function $f_0(x) + g_0(y)$ (the constants of integration). On the other hand, without knowing the payoffs, the direction of $D_x s_y$ globally is enough to determine the level sets of $s_y(\cdot, y)$ for each y .

Now, the set of continuous functions with the same level sets as a given function is exactly the set of monotonic transforms of that function. In other words, a function $G(\cdot)$ has the same level sets as $\frac{\partial s}{\partial y}(\cdot, y)$ if and only if:

$$G(x) = H\left(\frac{\partial s(x, y)}{\partial y}\right)$$

for some monotonic H , possibly depending on y . Fixing $y_0 \in Y$, we conclude that if \bar{s} is the surplus generating the given iso-husband sets, then another non-degenerate surplus s generates the same iso-husband sets if and only if there exists a function $H(z, y)$ with $H_z > 0$ such that:

$$s(x, y) = s(x, y_0) + \int_{y_0}^y H\left(\frac{\partial \bar{s}(x, t)}{\partial y}, t\right) dt.$$

This conclusion is implied by the following lemma.

Lemma 19 (Any couple can be smoothly matched) *Fix open sets $X \subset \mathbf{R}^m$, $Y \subset \mathbf{R}$ and $s \in C^{r+1}(X \times Y)$ non-degenerate, with $r \geq 1$.*

Given $(x', y') \in X \times Y$, and arbitrary small neighbourhoods U of x' and V of y' , we can find an atomless probability measure ν on V such the stable match $F \in C^r(\bar{U})$ between ν and the uniform measure μ on U satisfies $y' = F(x')$, and moreover: (31) holds for all $x \in \bar{U}$, and the husband's payoff v is a quadratic function on Y .

Proof. Since the lemma concerns only the behaviour of s near (x', y') , it costs no generality to assume X and Y bounded and $s \in C^{r+1}(\bar{X} \times \bar{Y})$. Take $v(y)$ to be a quadratic function of y satisfying

$$v''(y) = \text{const} > \max_{(x,y) \in \bar{X} \times \bar{Y}} s_{yy}(x, y) \quad (32)$$

and $v'(y') = s_y(x', y')$. Then strict concavity shows for each $x \in \bar{X}$,

$$u(x) = \max_{y \in \bar{Y}} s(x, y) - v(y) \quad (33)$$

is uniquely attained by some $y \in \bar{Y}$, denoted $y = F(x)$. Strict concavity also shows that if

$$s_y(x, y_0) - v'(y_0) = 0 \quad (34)$$

for some $y_0 \in \bar{Y}$ then $y_0 = F(x)$. In particular, $y' = F(x')$, so noting (32), the implicit function theorem shows $y_0(x) = F(x)$ is a C^r solution to (34) on some neighbourhood \bar{U} of x' . Differentiating (34) shows (31) holds on \bar{U} , so non-degeneracy of s implies $DF(x)$ is non-vanishing. Now we can take U and $V = F(U)$ to be as small as we please around x' and y' . We claim F is the stable match between the uniform measure μ on U , and its image $\nu := F_{\#}\mu$, while u and v above are the corresponding payoffs. The definition (33) shows (u, v) to be stable (3), hence integrating $u(x) + v(F(x)) = s(x, F(x))$ against μ shows $\gamma = (id \times F)_{\#}\mu$ attains the maximum and (u, v) attain the minimum in (9). Finally, ν is atomless since $DF \neq 0$ shows $F^{-1}(y)$ to be a C^r hypersurface in \bar{U} , hence μ negligible, by the implicit function theorem once more. ■

Remark 20 (Any couple has a strongly nested matching) *If $r \geq 2$ in the lemma above, then taking U smaller if necessary, so ∂U intersects each level set of F transversally (or at least \mathcal{H}^{m-1} negligibly), the co-area formula can be used as in [Chiappori, McCann & Pass (preprint)] to show $d\nu(y) = g(y)dy$ has a density satisfying $\log g \in C_{loc}(V)$. Replacing (X, Y) by (U, V) , Corollary 17 then shows the model is nested, since $v''(x) - s_{yy}(x, F(x)) > 0$.*

5 Applications

We now consider three applications of our framework.

5.1 Application 1: Income and fertility

Our first application considers a model introduced in [Low (2014)] and [Low (preprint)]. The main question relates to the impact of marriage market considerations on women’s decision to acquire higher education (and more precisely a post graduate degree). The trade-off Low considers is between human capital and fertility (what she calls ‘reproductive capital’). By engaging in post graduate studies, a woman increases her human capital, which boosts her future income (among other things). Most of the time, however, a post graduate degree will require postponing the birth of children to a later stage of her life, when her fertility may have declined. Husbands, in Low’s model, are interested in both the income of potential spouses and their fertility; the interaction between these two attributes, and their consequences on marital patterns are precisely what Low investigates.

In what follows, we generalize Low’s model in two directions. First, while Low assumes that fertility can take only two values (‘high’ for younger women and ‘low’ for older, more educated ones), we allow for general distributions of the two characteristics; in particular, we consider various correlation patterns between them (from independence to negative correlation). Second, the preferences we consider are slightly more general than Low’s, in the sense that we allow the presence of children to *decrease* utility when the family is poor; in other words, this is a model in which birth control technologies are either absent or largely imperfect (several examples can obviously be found in the developing world or in the history of Western societies). The second feature drastically changes the qualitative properties of the stable matching; specifically, the model is nested for some measures but not for others. Moreover, for specific measures (uniform in our case), both the nested and the unnested cases can be explicitly solved, which enable us to compare them in a systematic way.

5.1.1 The model

Men and women are parameterized by subsets $Y \subset \mathbf{R}$ and $X \subset \mathbf{R}^2$, respectively, where

- $y \in Y$ represents the husband’s income,
- $x_1 = p$ represents the wife’s fertility (probability of having a child)
- $x_2 = x$ is the wife’s income.

All couples with children receive some lump sum benefit B .¹²

¹²A similar example could readily be constructed with either means-tested benefits or tax rebates instead of a lump sum payment; the only difference being that the

Let q_i denote private expenditures of individual i and Q denote expenditures on children. If the couple has children, their preferences take the form

$$\begin{aligned} u_m(q_m, Q) &= q_m(Q + 1/2) \\ u_f(q_f, Q) &= q_f(Q + 1/2) \end{aligned}$$

whereas without children they are

$$u_g(q_g) = q_g, \quad g = m, f.$$

Suppose man y marries woman (x, p) . Then, with probability p , they have a child and maximize their combined utility $(q_m + q_f)(Q + 1/2)$ under the budget constraint:

$$q_m + q_f + Q = x + y + B$$

If we assume $x + y \geq 1/2 - B$, the solution to this maximization problem is

$$Q + 1/2 = q_m + q_f = \frac{x + y + B + 1/2}{2}$$

generating a total utility equal to $(x + y + B + 1/2)^2 / 4$. With probability $1 - p$, they do not have a child, and

$$u_m + u_f = q_m + q_f = x + y.$$

The total utility is therefore equal to the expected value of these two possible outcomes:

$$s(p, x, y) = p \frac{(x + y + B + 1/2)^2}{4} + (1 - p)(x + y).$$

In what follows, for the sake of simplicity we set the parameter B to $1/2$; the solution is therefore defined for all non negative x and y , and the surplus is

$$s(p, x, y) = p \frac{(x + y + 1)^2}{4} + (1 - p)(x + y).$$

A particular feature of this model is that if parents are poor enough (when $x + y < 1$, so that $Q < 1/2$) their utility with children is less than without - implying that no efficient birth control device is available. Such a situation can simply be ruled out by assuming that $x + y \geq 1$ for all couples; alternatively, we may consider cases in which some couples are 'poor' ($x + y < 1$) whereas others are 'wealthy' ($x + y \geq 1$). We consider both cases in our Examples 1 and 2 respectively; interestingly enough, the mathematical properties are quite different in the two settings, since one is nested whereas the other is not.

benefit would then be decreasing or increasing with the couple's income.

5.1.2 Solution

For this surplus, equation (15) becomes:

$$(x + y - 1)p = K(y). \quad (35)$$

where $K(y) = 2[\frac{dv(y)}{dy} - 1]$. Note, in particular, that if $K(\bar{y}) = 0$ for some \bar{y} then all women with $\bar{x} = 1 - \bar{y}$ marry \bar{y} irrespective of their p . In the explicit examples that follow, we will use the proportionate splitting condition to pin down $K(y)$. For now, we note two general properties:

- the iso-husband set of females (p, x) for husband y is given by the potential indifference curve

$$x = 1 - y + \frac{K(y)}{p}.$$

In the (p, x) plane, this curve is decreasing if and only if $K(y) > 0$. It intersects lines of constant p and x transversally, as well as the northwest and southeast corners. Since the other two corners match with the endpoints of Y , this will imply Z is empty for the rectangular domains considered in Examples 1 and 2.

- In addition, while

$$\frac{\partial^2 s}{\partial x \partial y} = \frac{p}{2} \geq 0 \quad (36)$$

we have that

$$\frac{\partial^2 s}{\partial p \partial y} = \frac{x + y - 1}{2}. \quad (37)$$

If $x + y \geq 1$, by (16) we expect the level sets of F to be decreasing curves in the (x, p) plane (that is, we expect $K(y) > 0$), meaning husbands face a trade-off of the income versus the fertility of their spouse. In the opposite case $x + y < 1$, these iso-husband curves may be increasing; we shall see an illustration in our second example.

Depending on the measures μ and ν , the problem may or may not admit a closed form solution. In what follows, we consider three examples. First, we provide a complete resolution when the distributions are uniform on $[0, 1] \times [1/2, 1]$ and $[1/2, 1]$, respectively. As a second example, we consider uniform distributions on $[0, 1] \times [0, 1]$ and $[0, 1]$.

Lastly, we again take husbands to be uniformly distributed on $[1/2, 1]$, but take the distribution of wives to be uniform on $[1/2, 1] \times [1/2, 3/4] \cup [0, 1/2] \times [3/4, 1]$. This distribution is chosen to reflect an anticorrelation

due to age between fertility and income. Fertility is certainly negatively correlated with age; on the other hand, women who pursue higher education tend to enter the marriage market later in life, and so we expect age and education to exhibit a positive correlation.

5.1.3 Example 1: uniform and ‘large’ incomes

We start with a benchmark case in which the distributions μ and ν are uniform on $[0, 1] \times [1/2, 1]$ and $[1/2, 1]$ respectively. Although the surplus degenerates at one corner $(p, x, y) = (0, \frac{1}{2}, \frac{1}{2})$ of $X \times Y$, Proposition 21 shows this example is nested, in view of Corollary 18. Moreover, the optimal matching function $F(x, p)$ can be solved for almost explicitly; we have the following result, proved in Appendix A; recall that y splits the population proportionately at (p, x) if

$$\begin{aligned} \mu\left(X_{\leq}(y, s_y(p, x, y))\right) &= \mu\left(\{(\bar{p}, \bar{x}) \mid \frac{\partial s(p, x, y)}{\partial y} \geq \frac{\partial s(\bar{p}, \bar{x}, y)}{\partial y}\}\right) \\ &= \nu\left([\frac{1}{2}, y]\right). \end{aligned}$$

Proposition 21 *Let μ and ν be uniform probability measures on $[0, 1] \times [1/2, 1]$ and $[1/2, 1]$, respectively. For each $(p, x) \in [0, 1] \times [1/2, 1]$ there is a unique $y \in [1/2, 1]$ that splits the population proportionately at (p, x) , and the optimal map takes the form $F(p, x) = y$.*

Note that, by the proof of this Proposition, we can actually derive an explicit formula for the optimal map:

$$F(p, x) := \sup_y \left\{ y \mid \mu\left(X_{<}(y, s_y(p, x, y))\right) > \nu\left([\frac{1}{2}, y]\right) \right\}.$$

We can go one step further, and identify the iso-husband curve consisting of the set of points (p, x) which match with a fixed y ; this is exactly the potential indifference curve (35), for the correct $K(y)$. As is clear from the proof of Proposition 21, for $y \in [1/2, \frac{e}{2(e-1)}]$, we have,

$$K(y) - \frac{y - 1/2}{\ln\left(\frac{y}{y-1/2}\right)} = 0.$$

For $y \in [\frac{e}{2(e-1)}, 1]$, we have $K(y) = \bar{x} + y - 1$, where \bar{x} is the unique solution in the interval $[1/2, 1]$ of the equation $x - y + (x + y - 1) \ln(\frac{y}{x+y-1}) = 0$. A plot of $K(y)$ is provided in Figure 1a, while Figure 1b gives the iso husband curves for various values of y . Notice $K'(y) = \bar{x}'(y) + 1$ diverges like $1/\log y$ at the endpoint $y = 1$ where the length of the iso-husband curve shrinks to zero; the payoff $v(y)$ therefore displays a comparable singularity in its second derivative.

Insert Figures 1a and 1b about here

5.1.4 Example 2: uniform, smaller incomes

We now consider the same surplus with different measures; namely, the marginals are uniform on $[0, 1] \times [0, 1]$ and $[0, 1]$ respectively. From a mathematical perspective this case provides an interesting illustration of an un-nested setting. Formally, our method to derive the optimal match fails in that case; it turns out that there are more than one y that split the population proportionately for certain choices (x, p) . However, we are able to use a refinement of the argument to obtain the explicit optimal match. Specifically, we use the symmetry embedded in the model to show that the optimal map is given by:

$$G(p, x) = \begin{cases} F(p, x) & \text{if } x > 1/2, \\ 1 - F(p, 1 - x) & \text{if } x < 1/2, \end{cases} \quad (38)$$

where F is as in the preceding Proposition. This solution displays a discontinuity along the line $x = \frac{1}{2}$; each wife $(p, \frac{1}{2})$ with income $x = \frac{1}{2}$ is indifferent between two distinct husbands, one richer ($y = \frac{e^{\frac{1}{p}}}{2(e^{\frac{1}{p}} - 1)}$) and the other poorer ($y = 1 - \frac{e^{\frac{1}{p}}}{2(e^{\frac{1}{p}} - 1)} = \frac{e^{\frac{1}{p}} - 2}{2(e^{\frac{1}{p}} - 1)}$). Although the total surplus generated by the latter marriage is smaller, her share of it remains the same. The corresponding isohusband curves are plotted on Figure 2a. We have also numerically solved this case; the resulting solution agrees with our theoretical solution and is graphed below (Figure 2b).

Insert Figures 2a and 2b about here

5.1.5 Example 3: anticorrelated marginals

In this case, the analytic solution is very complicated, and we provide only a numerical simulation. For $\epsilon = 0$ the numerical solution is plotted below; one interesting feature is that lower income men (whose isohusband curves lie in the green, yellow or orange regions) are indifferent between a selection of wives in both the low fertility, high income regime, $[0, \frac{1}{2}] \times [\frac{3}{4}, 1]$, and the high fertility, low income region $[\frac{1}{2}, 1] \times [\frac{1}{2}, \frac{3}{4} + \epsilon]$, whereas the highest income men, whose iso-husband curves lie in the dark red region, match exclusively with wives in the high fertility, low

income regime. This model does not fit the hypotheses of Theorem 16 when $\epsilon = 0$, since X is neither connected nor Lipschitz, but this shortcoming can be rectified by taking $\epsilon > 0$ arbitrarily small.

Insert Figure 3 about here

5.2 Application 2: A competitive Rochet-Choné model

In our second example, we revisit a seminal model of the literature on bidimensional adverse selection, due to [Rochet & Choné (1998)]. We therefore consider a hedonic model with:

- An n -dimensional space of products: $z = (z_1, \dots, z_n) \in Z \subset \mathbf{R}_+^n$;
- An n -dimensional space of buyers: $x = (x_1, \dots, x_n) \in X \subset \mathbf{R}_+^n$. Consumers are distributed on this space according to a probability measure μ . Each consumer will buy exactly one good. Their utility for purchasing product z for price $P(z)$ is given by $U(x, z) - P(z)$ where

$$U(x, z) = \sum_{i=1}^n x_i z_i,$$

- A one dimensional space of producers: $y \in Y \subset \mathbf{R}_+$. Producers are distributed according to a probability measure ν . Each producer will produce and sell exactly one good. Their profit in selling a good z for price $P(z)$ is $P(z) - c(y, z)$, where

$$c(y, z) = \frac{1}{2y} \sum_{i=1}^n z_i^2$$

is the production cost for producer y to produce a good of type z .

We note that this is exactly the model of [Rochet & Choné (1998)], with the additional twist that the monopoly producer is replaced with a competitive set of heterogeneous producers (whose productivity increases in y).¹³ As we will see, however, introducing competition has striking consequences. In particular, and somewhat surprisingly, there is no bunching in this new variant of their model: consumers of different types always buy goods of different types. We illustrate these facts with an example, finding explicitly the optimal matching for certain choices of μ and ν , and then provide a general proof.

¹³Clearly, heterogeneity is not crucial, since the measure on the set Y could be a Dirac. The crucial distinction with Rochet-Choné is the competitive assumption.

5.2.1 Equilibrium

As argued above, this hedonic pricing problem is equivalent to the matching problem on $X \times Y$ associated with the surplus:

$$s(x, y) = \max_{z \in Z} \left(\sum_{i=1}^n x_i z_i - \frac{1}{2y} \sum_{i=1}^n z_i^2 \right)$$

The maximum in the preceding equation is obtained (assuming $\{yx \mid y \in Y, x \in X\} \subset Z$) when

$$z_i = x_i y$$

and so

$$s(x, y) = \frac{1}{2} y \left(\sum_{i=1}^n x_i^2 \right).$$

Now, note that the twist condition is satisfied here; in fact, this surplus function is of index form, $s(x, y) = S(I(x), y) := \frac{I(x)y}{2}$, where $I(x) = |x|^2$, and S satisfies the classical Spence-Mirrlees condition $\frac{\partial^2 S}{\partial I \partial y} > 0$. The isoproducer curves take the form $\sum_{i=1}^n x_i^2 = K(y)$, and the matching is monotone increasing between $|x|$ and y .

5.2.2 An explicit example

We explicitly solve this problem for a particular choice of measures. Let $m = 2$ and μ be uniform measure (normalized to have total mass 1) on the quarter disk

$$\{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$$

and ν uniform on $[1, 2]$. The stable matching balances the mass of the quarter disc

$$\{x \mid |x|^2 \leq K(y)\}$$

with the ν mass of $[1, y]$. This implies

$$\frac{\pi 4K(y)^2}{4\pi} = y - 1$$

of $K(y) = \sqrt{y - 1}$. In other words, the optimal matching takes the form

$$F(x) = |x|^2 + 1.$$

Agent x then buys the product z such that:

$$z_i = x_i \left(\sum_{k=1}^n x_k^2 + 1 \right), \quad i = 1, \dots, n.$$

The envelope condition implies that agent x 's utility satisfies

$$\frac{\partial u}{\partial x_i}(x) = \frac{\partial s}{\partial x_i}(x, F(x))$$

or,

$$\frac{\partial u}{\partial x_i}(x) = x_i F(x) = x_i \left(1 + \sum_{k=1}^n x_k^2 \right).$$

Similarly,

$$v'(y) = \frac{1}{2} \sum_{i=1}^n x_i^2 = \frac{y-1}{2}.$$

Integrating these equations yields

$$u(x) = A + \frac{1}{2} \sum_{i=1}^n x_i^2 + \frac{1}{4} \left(\sum_{i=1}^n x_i^2 \right)^2$$

$$v(y) = B + \frac{(y-1)^2}{4}.$$

Now, for $y = F(x)$, we must have $u(x) + v(y) = s(x, y)$. Inserting $y = F(x) = |x|^2 + 1$ into our equation for v yields

$$u(x) + v(F(x)) = A + B + \frac{1}{2}|x|^2 + \frac{1}{4}|x|^4 + \frac{1}{4}|x|^4 = A + B + \frac{1}{2}|x|^2 + \frac{1}{2}|x|^4.$$

On the other hand,

$$s(x, F(x)) = \frac{1}{2}(|x|^2 + 1)|x|^2;$$

equating these implies $A = -B$. In what follows we assume $A = B = 0$; the interpretation is that the firm with the highest production costs ($y = 1$) makes zero profit.

We now turn our attention to the equilibrium pricing schedule, $P(z)$. Following [Chiappori, McCann & Nesheim (2010)], the stability conditions

$$u(x) + v(y) \geq s(x, y) \geq U(x, z) - c(y, z)$$

imply

$$\inf_y (v(y) + c(y, z)) \geq P(z) \geq \sup_x (U(x, z) - u(x))$$

or

$$\inf_y \left(\frac{(y-1)^2}{4} + \frac{1}{2y} \sum_{i=1}^n z_i^2 \right) \geq P(z) \geq \sup_x \left(\sum_{i=1}^n x_i z_i - \frac{1}{2} \sum_{i=1}^n x_i^2 - \frac{1}{4} \left(\sum_{i=1}^n x_i^2 \right)^2 \right).$$

The solution takes the form

$$P(z) = \bar{P}(Z),$$

where $Z = |z|^2$. Note that when agent y sells good z , we must have

$$P(Z) = \frac{(y-1)^2}{4} + \frac{1}{2y} \sum_{i=1}^n z_i^2 = \frac{(y-1)^2}{4} + \frac{1}{2y} Z.$$

The first order conditions then imply

$$\frac{1}{2}(y-1) = \frac{1}{2y^2} Z,$$

so that $Z = y^2(y-1)$. We can then solve for $P(Z)$ in terms of y ;

$$\begin{aligned} P &= \frac{(y-1)^2}{4} + \frac{1}{2y} Z \\ &= \frac{1}{4} (3y-1)(y-1). \end{aligned}$$

Noting that $Z = |z|^2 = y^2|x|^2 = y^2(y-1)$ then yields a parametric representation for the curve $(Z, P(Z))$:

$$(Z, P(Z)) = \left(y^2(y-1), \frac{1}{4} (3y-1)(y-1) \right)$$

This is plotted on Figure 4.

Insert Figure 4 about here

Some comments can be made on this solution. First, and unlike the Rochet-Choné monopoly case, there is no *exclusion*; all agents buy products. This property is easily understood from the matching formulation; as every potential match generates a positive surplus, surplus maximization implies that no agents will be excluded.¹⁴ Note, however, that the introduction of outside options might reverse this conclusion. More interesting (and more surprising) is the second finding, namely that *there is no bunching of any type*: the efficient allocation is fully separating so that different agents *always* buy different goods. This no bunching conclusion is less intuitive; it is not clear, a priori, whether it holds for general choices of the measures μ and ν , or it is an artifact of the particular measures we use in this example. We now present a theorem which states that the former interpretation is correct.

¹⁴Exclusion is linked to the monopoly logic: by excluding some consumers, the monopolist can increase the rent received from other buyers.

5.2.3 The general case

The method for deriving the optimal matching will always work here; as the surplus has an index form, issues with level sets crossing do not arise. Of particular interest, the no bunching result is completely general, as the following result, proved in an appendix, confirms:

Theorem 22 *Different consumer types always buy different goods. That is, if $x \neq \bar{x}$, and z and \bar{z} represent the respective goods they choose, then $z \neq \bar{z}$.*

Proof. See Appendix ■

It can be stressed that there are no restrictions on the measures in this theorem. In particular, it applies when μ is uniform measure on the unit square in \mathbf{R}^2 , and ν is a Dirac mass at $y = 1$; these conditions provide the closest analogue to the Rochet-Choné problem. The difference is that here we have a continuum of homogeneous producers (each with identical production cost $\frac{|z|^2}{2}$ for good $z \in \mathbf{R}^2$) competing with each other, rather than a monopoly. In other words, the bunching phenomena present in the Rochet-Choné solution are not intrinsically linked to the multidimensionality of the consumer type space; rather, they are due to market inefficiencies created by the monopoly.

Finally, it is interesting to note that the framework under consideration can always be seen as a model of competition under adverse selection; in that sense, the matching approach provides a natural definition of an equilibrium in such a framework. A crucial remark, however, is that the model is characterized by its *private value* nature, since the producer's profit is not directly related to the identity of the consumer buying its product (it only depends on the characteristics of the product and its price). This is in sharp contrast with a common value setting, in which the buyer's characteristics directly impact the producer's profit. Think, for instance, of an insurance model à la [Rothschild & Stiglitz (1976)], in which the same insurance contract generates different profit depending on the buyer's unobserved characteristics (in that case her risk). Technically, the producer's cost function c now depends on y and z as before, but also on x . The key point is that, in such a common value context, the equivalence between matching and hedonic models is lost. In particular, it is not true in general that there always exists a price function $P(z)$ such that a stable matching can be implemented as an hedonic equilibrium. Our approach, therefore, does construct a first bridge between the literatures on matching and optimal transportation on the one hand, and *competition* under asymmetric information on the other hand; however, the relationship is specific to

private value models. In the non-competitive setting of the principal-agent framework, the analogous connection found in [Carrier (2001)] and [Figalli, Kim & McCann (2011)] has proven extremely fruitful.

5.3 Application 3: a simple hedonic model

As a last illustration, we consider a simple hedonic model, in which consumers have heterogeneous tastes for differentiated products, which are produced by firms with heterogeneous cost functions. The basic framework is similar to the previous, Rochet-Choné one¹⁵, except for one feature: the cost function is now:

$$c(y, z) = \sum_{i=1}^n \frac{z_i^2}{2y_i}$$

where the vector $y = (y_1, \dots, y_n)$ is producer-specific. The multidimensionality of y reflects the fact that different producers made have different comparative advantages in producing *some* of the characteristics but not others; for instance, a producer may be good at producing fast cars but less efficient for smaller ones.¹⁶ Note that, unlike most of the empirical IO literature (starting with the seminal contributions by [Berry, Levinsohn & Pakes (1995)] and [Berry, Levinsohn & Pakes (2004)]), which assume imperfect competition, our hedonic framework posits that producers are price takers. In that sense, our matching framework can be seen as an alternative approach to modeling competition in differentiated products.

The characterization of the equilibrium follows the same path as before. The surplus function is:

$$s(x, y) = \max_{z \in Z} \sum_{i=1}^n \left(x_i z_i - \frac{z_i^2}{2y_i} \right).$$

Here, the maximum is obtained when $z_i = x_i y_i$, leading to:

$$s(x, y) = \frac{1}{2} \sum_{i=1}^n x_i^2 y_i$$

Since s is continuous, existence follows from Theorem 4. Moreover, this form is a particular case of the general structure studied in subsection 2.4. It follows that the twist condition is satisfied whenever

¹⁵In particular, we keep utilities of the form $\sum_i x_i z_i$, where $x = (x_1, \dots, x_n)$ denotes an agent's idiosyncratic valuations of product characteristics. For empirical applications, x will typically be considered as a random vector.

¹⁶We maintain here the assumption that each firm produces one good. This can readily be relaxed provided that production functions are linear in *quantity* produced (while quadratic in characteristics z).

$x \in \mathbf{R}_+^n$, an assumption we maintain throughout the current subsection. This guarantees uniqueness and purity:¹⁷ there exists a function F such that x is matched with $y = F(x)$. The simplicity of these existence and uniqueness arguments is indeed an important advantage of the matching approach.

As mentioned above, purity implies that different agents are matched with different producers. Moreover, since $z_i = x_i y_i$, a result by Lindentraub implies that different agents always buy different products; in other words, there is no bunching in this model. Clearly, these conclusions follow from the assumption that the dimension of heterogeneity is the same for individuals and firms. If this assumption is relaxed (for instance by assuming that $n < m$), then a continuum of different buyers purchase from the same producer, and the market shares can be analyzed using the results in Section 3.

The specific form of function F depends on the distributions of individual and firm characteristics. As previously, we illustrate our approach by explicitly solving this problem for a particular choice of measures. Let μ be the uniform measure (normalized to have total mass 1) on the disk $\sum_{i=1}^n (x_i - a_i)^2 \leq 1$ and let ν be the uniform measure (again normalized to have total mass 1) on the disk $\sum_{i=1}^n (y_i - b_i)^2 \leq 1$ with $a_i, b_i > 1, i = 1, \dots, n$. Then the support of the stable measure is born by the function $F = (F_1, \dots, F_n)$:

$$y_i = F_i(x) = x_i - a_i + b_i$$

Agent x then buys the product z such that:

$$z_i = x_i(x_i - a_i + b_i)$$

and firm y produces the product z such that:

$$z_i = (a_i - b_i + y_i) y_i$$

Utilities are derived as before:

$$\begin{aligned} \frac{\partial u}{\partial x_i}(x) &= \frac{\partial s}{\partial x_i}(x, F(x)) \\ &= x_i(x_i - a_i + b_i) \end{aligned}$$

¹⁷In fact, the restriction of x to the positive orthant is not essential, since the twist condition holds for all x in the complement of the coordinate axes. The latter form a set of measure zero which it is always possible to omit, as in [Chiappori, McCann & Nesheim (2010)].

therefore

$$u(x) = K + \sum_{i=1}^n \left(\frac{x_i^3}{3} - \frac{a_i - b_i}{2} x_i^2 \right)$$

and similarly

$$v(y) = K' + \frac{1}{6} \sum_{i=1}^n (a_i - b_i + y_i)^3$$

with $K + K' = 0$.

The hedonic price satisfies:

$$\inf_y (v(y) + c(y, z)) \geq P(z) \geq \sup_x (U(x, z) - u(x))$$

which gives:

$$P(z) = -K + \frac{1}{12} \sum_{i=1}^n \left[\left((a_i - b_i)^2 + 4z_i \right)^{\frac{3}{2}} + (a_i - b_i) \left((a_i - b_i)^2 + 6z_i \right) \right].$$

As usual in a quasi-linear framework (i.e. in the absence of income effects), the price schedule is defined up to an additive constant K ; the latter can be pinned down, for instance, by imposing conditions on the profit of the least productive firm.

6 Conclusion

This paper provides a general characterization of multidimensional matching models, in terms of existence, uniqueness and qualitative properties of stable matches. We show that recent developments in the literature on multidimensional optimal transport can be exploited to derive conditions under which stable matches are unique and pure, and to understand their local geometry. We consider a first application of such models to a competitive version of a standard, hedonic model in which heterogeneous consumers purchase differentiated commodities from heterogeneous producers.

Of specific interest are situations in which the dimensions of heterogeneity on the two sides of the market are unequal. We explore the topology of the ‘indifference sets’ that arise in this setting, and provide conditions under which they can be expected to be smooth manifolds of dimension $m - n$. In particular, we investigate the set of ‘multi-to-one dimensional matching problems’, and we introduce a nestedness criterion under which the equilibrium match can be found more or less explicitly. Lastly, we stress the deep relationships that exist between the multi-to-one dimensional framework and models of competition under multidimensional asymmetric information, at least in the case of private

values. Considering a competitive variant of the seminal, Rochet-Choné (1998) problem, in which goods can be produced by a 1-dimensional, heterogeneous distribution of producers, rather than a single monopolist, we provide a full characterization of the resulting equilibrium price schedule. In particular, we show that in our competitive framework, and in contrast to the original monopolist setting, there is never bunching; that is, consumers of different types *always* buy goods of different types.

Appendices

Appendix A Verification of Examples 1 and 2

We begin with Proposition 21.

Proof. We verify that for each $(p, x) \in [0, 1] \times [1/2, 1]$, there is a unique $y \in [1/2, 1]$ splitting the population proportionally at (p, x) ; the result then follows from Corollary 18.

The proportionally splitting level curves $X(y, k(y))$ take the form $x^y(p) = 1 - y + \frac{K(y)}{p}$, where $K(y)$ is chosen to satisfy the population splitting condition. Consider now the way that the curves $X(y, k(y))$ intersect the boundary; as each $p \mapsto x^y(p)$ is continuous on $(0, 1)$ and monotone decreasing (note that $K(y) = (x + y - 1)p \geq 0$), it intersects either $[0, 1] \times \{1/2\}$ in a unique point $(p, x) = (p(y), 1/2)$ or $\{1\} \times [1/2, 1]$ in a unique point $(p, x) = (1, x(y))$. We will prove the following two properties:

1. $K(y)$ is monotone increasing in y .
2. The boundary intersection points have a certain monotonicity property. Precisely, for each $y_0 < y_1$, one of the following occurs:
 - (a) $X(y_0, k(y_0))$ intersects $[0, 1] \times \{1/2\}$ and $X(y_1, k(y_1))$ intersects $\{1\} \times [1/2, 1]$.
 - (b) $X(y_0, k(y_0))$ and $X(y_1, k(y_1))$ both intersect $[0, 1] \times \{1/2\}$, and $p(y_0) < p(y_1)$.
 - (c) $X(y_0, k(y_0))$ and $X(y_1, k(y_1))$ both intersect $\{1\} \times [1/2, 1]$, and $x(y_0) < x(y_1)$.

These two facts will imply the desired result as follows: 1) will imply that $x^{y_1}(p) - x^{y_0}(p)$ is decreasing in p , for fixed $y_0 < y_1$, as the derivative of this function is $x^{y_1}(p) - x^{y_0}(p) = \frac{K(y_0) - K(y_1)}{p^2} < 0$. This means that if two of the population splitting curves intersect, (that is $x^{y_1}(\bar{p}) - x^{y_0}(\bar{p}) =$

0) within the given domain, then $x^{y_1}(p) < x^{y_0}(p)$ for all $p > \bar{p}$. This in turn implies that the boundary intersection points satisfy one of the following:

- a) $X(y_1, k(y_1))$ intersects $[0, 1] \times \{1/2\}$ and $X(y_1, k(y_1))$ intersects $\{1\} \times [1/2, 1]$.
- b) $X(y_0, k(y_0))$ and $X(y_1, k(y_1))$ both intersect $[0, 1] \times \{1/2\}$, and $p(y_0) > p(y_1)$.
- c) $X(y_0, k(y_0))$ and $X(y_1, k(y_1))$ both intersect $\{1\} \times [1/2, 1]$, and $x(y_0) > x(y_1)$.

This clearly contradicts point 2) above and so establishes the desired result.

To complete the proof, then, it remains only to verify points 1) and 2). We first consider population splitting curves which pass through $[0, 1] \times \{1/2\}$. In this case, the proportional splitting condition is given by

$$\begin{aligned} y - 0.5 &= \int_{0.5}^1 \frac{p(y)(\bar{x} + y - 1)}{(x + y - 1)} dx \\ &= (\bar{x} + y - 1)p(y)[\ln(1 + y - 1) - \ln(0.5 + y - 1)] \\ &= (y - 0.5)p(y) \ln\left(\frac{y}{y - 0.5}\right). \end{aligned}$$

This means that $p(y) = \frac{1}{\ln(\frac{y}{y-0.5})}$. A simple calculation shows that this function is increasing in y . The function can also be inverted to obtain $y(p) = \frac{e^{\frac{1}{p}}}{2(e^{\frac{1}{p}} - 1)}$; this tells us that the population splitting level curves pass through $[0, 1] \times \{1/2\}$ precisely for $y \leq y(1) = \frac{e}{2(e-1)}$. For y in this region, the monotonicity of $p(y)$ implies that 2) b) holds, and we note that $K(y) = \frac{y-1/2}{\ln(\frac{y}{y-1/2})}$, which is also monotone increasing, verifying 1) in this region.

Therefore, for $y \geq \frac{e}{2(e-1)}$, the curve $X(y, k(y))$ intersects $\{1\} \times [1/2, 1]$. This confirms part 2) a); to complete the proof, it remains only to show that $x(y)$ and $K(y)$ are monotone increasing on $[\frac{e}{2(e-1)}, 1]$. In this region, the proportional splitting condition gives us:

$$\begin{aligned} 1 - y &= \int_{x(y)}^1 1 - \frac{1(x(y) + y - 1)}{(x + y - 1)} dx \\ &= 1 - x(y) - (x(y) + y - 1)[\ln(1 + y - 1) - \ln(x(y) + y - 1)] \end{aligned}$$

or, equivalently, $x(y)$ is the unique solution in $[0.5, 1]$ of

$$0 = f(x, y) := x - y + (x + y - 1) \ln\left(\frac{y}{x + y - 1}\right). \quad (39)$$

Differentiating implicitly, we have

$$x'(y) = -\frac{f_y}{f_x}. \quad (40)$$

We have, for $x < 1$

$$f_x = 1 + \ln\left(\frac{y}{x + y - 1}\right) - 1 = \ln\left(\frac{y}{x + y - 1}\right) > 0$$

(as, for $x < 1$, $\frac{y}{x+y-1} > 1$).

We now show $f_y < 0$. We have

$$f_y = -2 + \ln\left(\frac{y}{x + y - 1}\right) + \frac{x + y - 1}{y}. \quad (41)$$

Now, as $x \leq 1$, the last term satisfies

$$\frac{x + y - 1}{y} \leq 1 \quad (42)$$

(with equality only when $x = 1$), and the second to last term is decreasing in y , so it is less than its value at $y = \frac{e}{2(e-1)}$:

$$\ln\left(\frac{y}{x + y - 1}\right) \leq \ln\left(\frac{e}{2(e-1)}\right) - \ln\left(\frac{e}{2(e-1)} + x - 1\right) \quad (43)$$

$$\leq \ln\left(\frac{e}{2(e-1)}\right) - \ln\left(\frac{e}{2(e-1)} - \frac{1}{2}\right) \quad (44)$$

$$= 1, \quad (45)$$

where the second inequality follows, as $x \geq \frac{1}{2}$. Note that we have equality above only if $y = \frac{e}{2(e-1)}$ and $x = \frac{1}{2}$.

Now note that one of the equalities (42) or (45) is always strict (as either $x < 1$ or $x > \frac{1}{2}$). Therefore, the derivative (41) is negative on the relevant range.

As $f_y < 0$ and $f_x > 0$, (40) tells us $x'(y) > 0$. Finally, we have $K(y) = x(y) + y - 1$, and so the monotonicity of $x(y)$ tells us $K(y)$ is also strictly increasing on this region. ■

Finally, we turn our attention to uniform measures on $[0, 1]$ and $[0, 1]^2$, and the proof of formula (38) in Section 5.1.4. The proof requires the following Lemma.

Lemma 23 *Assume μ is uniform on $[0, 1]^2$ and ν is uniform on $[0, 1]$. If (p, x, y) is in the support of the optimal matching γ , and $p > 0$ then either $x \geq \frac{1}{2}$ and $y \geq \frac{1}{2}$ or $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$.*

Proof. Note that $s(p, 1 - x, 1 - y) - s(p, x, y) = 2 - x - y$ exhibits no interaction terms. As the measures are symmetric under the transformation $(p, x) \rightarrow (p, 1 - x)$ and $y \rightarrow 1 - y$, the (unique) stable matching will be symmetric under the transformation $(p, x, y) \rightarrow (p, 1 - x, 1 - y)$.

Now, let (p, x, y) belong to the support of γ ; then, by invariance, we must also have $(p, 1 - x, 1 - y)$ in the support of γ . Applying 2 s monotonicity to these points yields

$$s(p, x, y) + s(p, 1 - x, 1 - y) \geq s(p, 1 - x, y) + s(p, x, 1 - y).$$

Now note that for fixed p , the function $(x, y) \mapsto s(x, y, p)$ satisfies the Spence-Mirrlees condition, $\frac{\partial^2 s}{\partial x \partial y} = \frac{p}{2} > 0$. This implies

$$(x - (1 - x))(y - (1 - y)) = (2x - 1)(2y - 1) \geq 0$$

which is equivalent to the stated result. ■

We now prove formula (38) in Section 5.1.4.

Proof. By the preceding Lemma, the optimal map maps the region $[0, 1] \times [\frac{1}{2}, 1]$ to $[\frac{1}{2}, 1]$ and $[0, 1] \times [0, \frac{1}{2}]$ to $[0, \frac{1}{2}]$. The map in the first region then must be given by $G(p, x) = f(p, x)$, and the mapping in the second region is $G(p, x) = 1 - f(p, 1 - x)$ by symmetry. ■

Appendix B Proof of no bunching

Here we prove the no bunching result, Theorem 22, for our competitive variant of Rochet and Choné's screening model.

Proof. We prove that $x \neq \bar{x}$ always buy different goods. Let y and \bar{y} be the buyers they match with, respectively, and assume that x and \bar{x} both buy the good z (from y and \bar{y} respectively). We will show this implies $x = \bar{x}$.

We have that $z = yx = \bar{y}\bar{x}$.

As (x, y) and (\bar{x}, \bar{y}) belong to the support of the optimal measure γ , they satisfy the 2-monotonicity condition:

$$s(x, y) + s(\bar{x}, \bar{y}) \geq s(x, \bar{y}) + s(\bar{x}, y)$$

Now, we have that $s(x, y) = x \cdot z - \frac{1}{2y}|z|^2$, $s(\bar{x}, \bar{y}) = \bar{x} \cdot z - \frac{1}{2\bar{y}}|z|^2$ and $s(x, \bar{y}) \geq x \cdot z - \frac{1}{2\bar{y}}|z|^2$, $s(\bar{x}, y) \geq \bar{x} \cdot z - \frac{1}{2y}|z|^2$. Substituting into the above equation yields:

$$x \cdot z - \frac{1}{2y} |z|^2 + \bar{x} \cdot z - \frac{1}{2\bar{y}} |z|^2 \geq s(x, \bar{y}) + s(\bar{x}, y) \geq x \cdot z - \frac{1}{2\bar{y}} |z|^2 + \bar{x} \cdot z - \frac{1}{2y} |z|^2$$

The right and left hand side are identical, so we must have equality throughout. In particular, we have that

$$s(\bar{x}, y) = \bar{x} \cdot z - \frac{1}{2y} |z|^2.$$

This implies $z = y\bar{x}$ (as z maximizes the joint surplus for y and \bar{x}). But we also have $z = yx$ from above. Therefore

$$y\bar{x} = yx.$$

Canceling the y gives the desired result. ■

Appendix C Figures

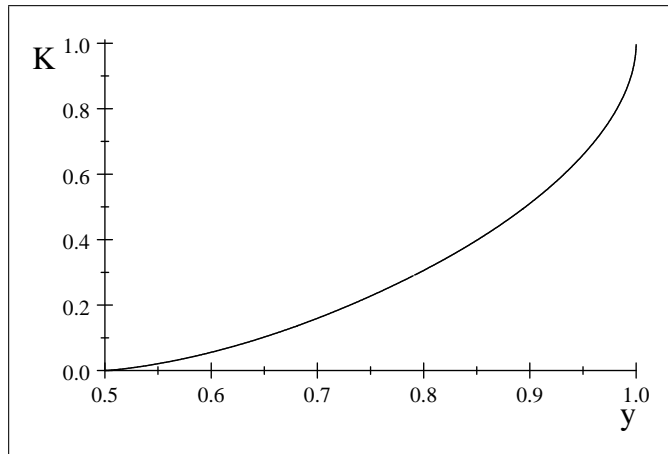


Figure 1a: $K(y)$

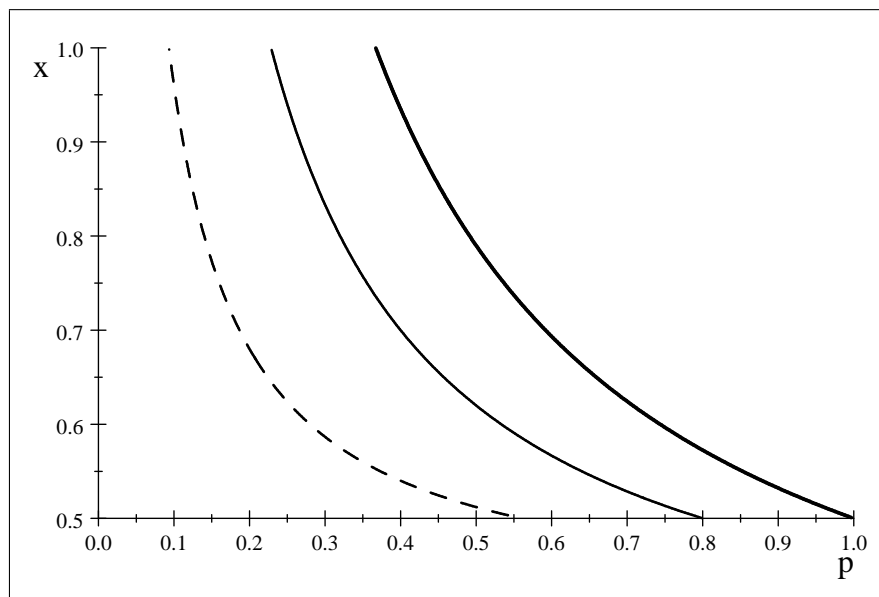


Figure 1b: Isohusband curves for $y = .6, .7, .8$

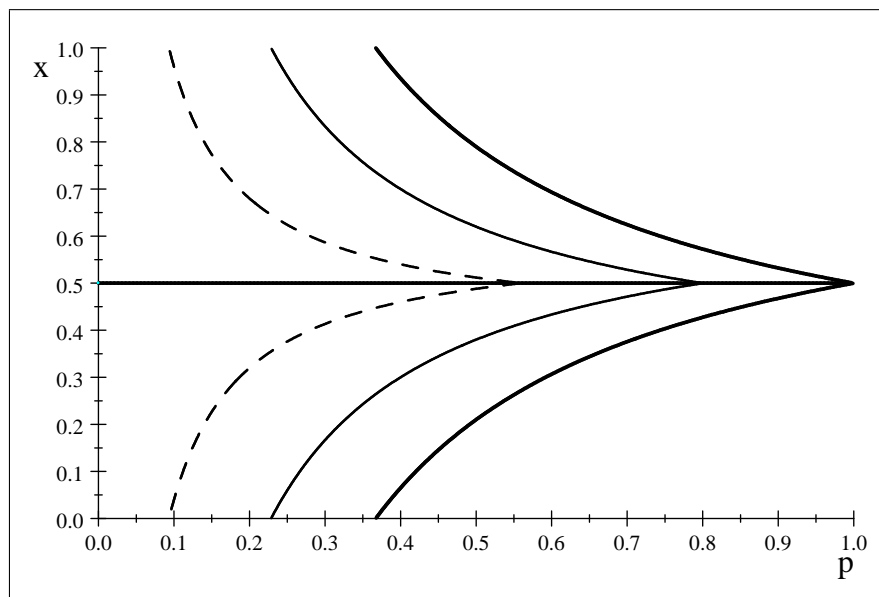


Figure 2a: Isohusband curves for $y = .2, .3, .4$ and $.6, .7, .8$

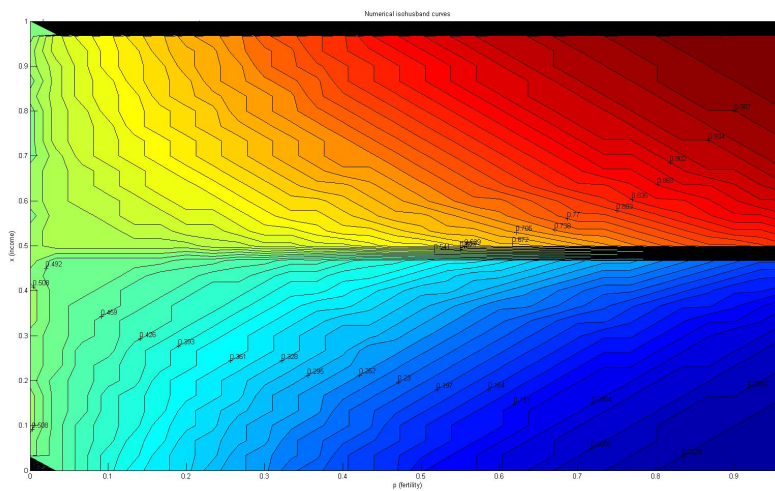


Figure 2b: Numerically generated iso-husband curves when both men's income and women's fertility and income uniformly distributed. The horizontal and vertical axes represent the fertility and income of women, respectively. The curves are iso-husband curves; that is, level curves of the matching function $F(p, x)$.

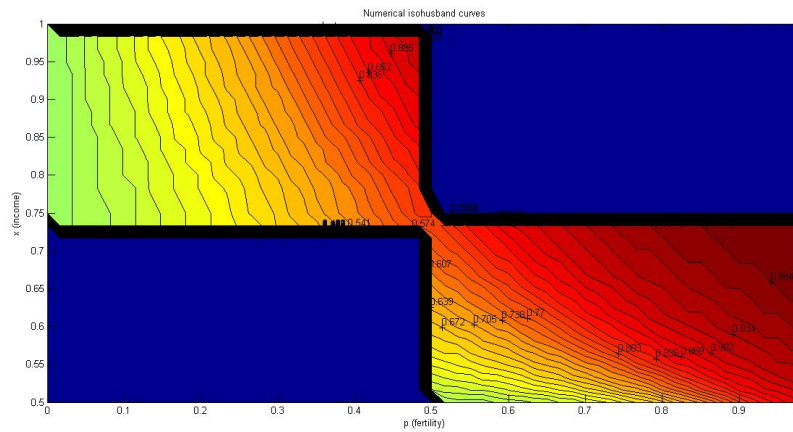


Figure 3: Numerically generated iso-husband curves when women's fertility and income are anti-correlated. The horizontal and vertical axes represent the fertility and income of women, respectively. The curves are iso-husband curves; that is, level curves of the matching function $F(p, x)$.

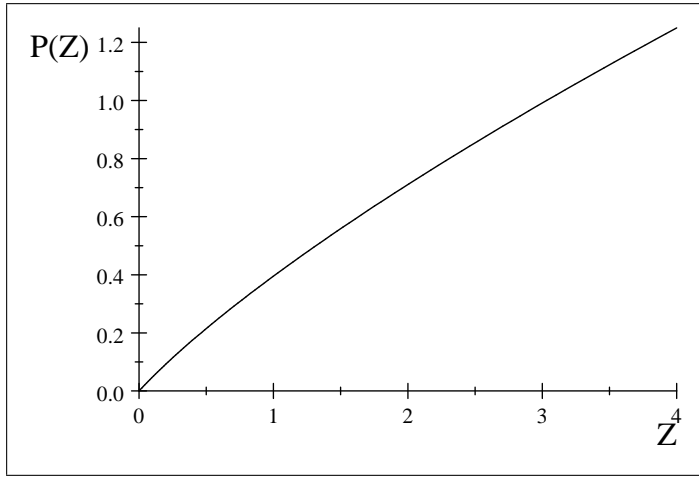


Figure 4: pricing schedule

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