Aggregate Fluctuations from Independent Sectoral Shocks: Self-Organized Criticality in a Model of Production and Inventory Dynamics*

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Abstract

This paper illustrates how fluctuations in aggregate economic activity can result from many small, independent shocks to individual sectors. The effects of the small independent shocks fail to cancel in the aggregate due to the presence of two non standard assumptions: local interaction between productive units (linked by supply relationships), and non-convex technology. We also argue that neither feature on its own would suffice. In the case of a simple model, we explicitly calculate the distribution of aggregate activity in the limit of an infinite number of independent disturbed sectors.

1 Introduction

Explaining the observed instability of economic aggregates is a long-standing puzzle for economic theory. A number of possible reasons for variation in the pace of production are easily given, such as stochastic variation in the timing of households' desired consumption of produced goods, or stochastic variation in the costs of production. But it is hard to see why there should be large variations in those factors that are synchronized across the entire economy – why most households should want to consume less at exactly the same time, or why most firms should find it an especially opportune moment to produce at the same time. Instead, it seems more likely to suppose that variations in demand or in production costs in different parts of the economy should be largely independent.

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Thus, one might ask, should one not expect these local variations to cancel out, for the most part, in their effects on the aggregate economy, due to the law of large numbers? Fluctuations in activity of macroeconomic significance, it might be thought, should occur only when many independent shocks happen by coincidence to have the same sign, and this should be an extremely unlikely event (with the probability of occurrence decreasing exponentially with the square of the size of the event, by the central limit theorem). [1]

The conventional response is that aggregate shocks are needed as the source of business cycles, i.e., large exogenous events that affect all parts of the economy in a similar way. Especially important candidates are changes in government policy that affect financial markets, and through them the entire economy, or that affect the budgets of many people in the economy at once. But the significance of changes in monetary and fiscal policy as a source of aggregate shocks has been challenged in much recent work. It is argued, for instance, that self-interested economic agents should respond to changes in either the government deficit or in the quantity of money in circulation in ways that neutralize the effects of these policy shocks (increasing saving to absorb increased issuance of government debt, varying the level of money prices to keep the real money supply unchanged), without requiring any change in the production and consumption of goods and services. As a result much work in business cycle theory in the last decade has instead emphasized aggregate shocks to the production technology or to household preferences; but these are types of disturbances which do not obviously have an aggregate character. [2]

An alternative approach suggested in some recent work proposes that economies possess intrinsically unstable dynamics, that even in the absence of external shocks would result in persistent deterministic fluctuations, such as a limit cycle, or even deterministic "chaos". [3] A problem with this type of model is that it implies that aggregate fluctuations should involve motion on a low-dimensional attractor. Yet analysis of economic time series has not revealed structure of this kind. In particular, statistical tests intended to measure the dimension of the attractor do not find evidence of a low dimension, at least insofar as the question can be settled with short time series of the kind available in economics. [4] The irregularity of economic time series would seem to require an explanation of another sort.

Here we pursue an alternative type of explanation. Our proposal is that the effects of many small independent shocks to different sectors of the economy do not cancel out in the aggregate, due to absence of the kind of linear aggregation of shocks required for the law of large numbers to apply. The conventional reasoning fails as a result of significantly nonlinear, strongly localized interactions

[1] For an important early discussion of this problem in economic theory, see Jovanovic (1987).

[2] The issue is sidestepped in most recent work in business cycle theory by the assumption of a single representative firm and a single representative household. See, e.g., Kydland and Prescott (1982). Long and Plosser (1983) consider a multi-sector model, but their model has the property that aggregate randomness disappears if the number of sectors is made large, for reasons of the kind that we sketch in section 3.

[3] For a survey of this literature, see Boldrin and Woodford (1990).

[4] For a survey of the empirical work on this issue, see Scheinkman (1990).

between different parts of the economy. The type of macroscopic instability that can result has been studied in variety of previous contexts, under the name of "self-organized criticality." We discuss applications of this idea in physics and elsewhere in the following section.

We illustrate the idea here in the context of an extremely simple model of a multi-sector, multi-stage production process. A large number of production units each buy goods from and sell goods to a small number of "neighboring" production units, using the goods they buy to produce the goods they sell. The significantly nonlinear interaction between "neighborin" units' decisions results from non-convexities in the production technology, that are important at the level of the production unit, though not on the scale of the aggregate economy. The exogenous shocks that drive the economy are independent fluctuations in flow demands for a large number of different types of final goods. In the limit as the number of sectors is made large, the aggregate flow demand for final goods becomes steady. Yet the resulting distribution of levels of aggregate production (appropriately scaled) converges to a Pareto-Levy distribution. [5] Hence the variability of aggregate production (as measured, for instance, by the ratio of the inter-quartile range to the median) does not vanish even in the limit of an unboundedly large number of sectors subject to independent shocks. Furthermore, not only is the limiting distribution not a constant, it is a distribution with the property that the probability of large events falls off only algebraically, rather than exponentially, with the size of the event. Thus very large fluctuations are predicted to occur surprisingly often. Both features of this result illustrate the important consequences of taking account of the nonlinearity of the interactions between neighboring units.

2 Self-Organized Critical Systems

The dynamics of large interactive systems has been much studied by physicists concerned with the properties of condensed matter. In a variety of contexts, physicists have noted the possibility of a "critical state", in which independent microscopic fluctuations can propagate so as to give rise to instability on a macroscopic scale. In a "subcritical" state, changes in one part of the system have a sufficiently weak effect upon neighboring parts that the state in different regions of the system is correlated only over short distances. The correlation falls off exponentially with distance, and if one looks at average behavior over a region that is large compared to the "correlation length" (inverse of the rate of decay of correlation with distance), spontaneous fluctuations are not observed. On the other hand, when some parameter of the system is "tuned" to an appropriate value, a "critical" state may be reached, in which the correlation between parts of the system ceases to decay exponentially with distance, and in which spontaneous macroscopic fluctuations may be observed in a system of arbi-

[5] For the classical discussion of the type of distribution we refer to, and the generalization of the central limit theorem upon which our results depend, see Levy (1925, 1954). For further discussion, and applications to economic data, see Mandelbrot (1960, 1963, 1964).

trary size. The spontaneous magnetization of a ferromagnetic material when its temperature drops to the Curie point is a classic example of the phenomenon. Beyond the critical point (*i.e.,* below the Curie point), a new structure forms that again does not exhibit macroscopic fluctuations, but at the critical point itself macroscopic fluctuations are possible in the absence of an external perturbation, and even arbitrarily small external perturbations can have large effects upon the macroscopic state (*e.g.,* imposition of a weak external magnetic field can determine the direction of polarization of the ferromagnet).[6]

The problem with this as a model of spontaneous macroeconomic instability is that, traditionally, critical states were thought to be associated with certain "critical" parameter values (such as the temperature in the example just mentioned), that would almost certainly not occur in any existing system unless they were "tuned" to be at the critical value in a laboratory experiment. [7] But more recently, it has been argued that large interactive dynamical systems can "self-organize" into a critical state (Bak, Tang, and Wiesenfeld, 1988). That is, the critical state can actually be an attractor for the dynamical system, toward which the system naturally evolves, and to which it returns after being perturbed by some large external shock.

The prototypical example of such "self-organized criticality" is a sand pile (Bak and Chen, 1991). When the slope of the pile is nowhere too steep, dropping on additional grains of sand at randomly chosen sites will have no macroscopic effects (though of course it modifies the shape of the pile in the immediate area in which a grain is dropped), as at most small numbers of grains will shift position in each case. However, randomly dropping on additional sand will result in the slope of the pile increasing to a critical slope, at which point avalanches of all sizes (limited only by the size of the pile) can occur in response to the dropping of a single additional grain of sand. A sandpile with a slope that is initially greater than the critical slope also evolves toward it, in this case through an immediate large avalanche that collapses the pile to a flatter and more stable configuration. The existence of the self-organized critical state is robust not only to perturbations of the initial shape of the pile, but to changes in the type of sand used (although differently shaped grains will change the value of the critical slope). This sort of robustness makes such a state a plausible model of spontaneous macroscopic instability in systems observed in nature.

Self-organized critical systems have been proposed as models of a variety of physical phenomena, including earthquakes (Bak and Tang, 1989), volcanic eruption (Diodati *et al.,* 1991), and turbulence. The greatest success of the theory thus far has been its explanation of the famous Gutenberg-Richter (1956) law for the size distribution of earthquakes. Applications to biological phenomena have also been proposed (Bak, Chen and Creutz, 1991; Kauffman and Johnson, 1991).

[6] See, *e.g.,* Stanley (1971). For an elementary discussion, see Prigogine (1980, chap. 6).

[7] Jovanovic's (1987) examples of economic models in which independent sectoral shocks produce aggregate fluctuations no matter how large the number of sectors, which do not depend upon the local interaction or non-linear interaction that we stress here, are special in exactly this sense.

Here we demonstrate the possible occurrence of a self-organized critical state as a result of factor demand linkages between sectors in a large economy. We begin with some general remarks about production scheduling and inventory dynamics, with particular emphasis upon the conditions under which independent sectoral variations in the flow of orders for final goods can give rise to sizeable fluctuations in aggregate production.

3 Sectoral Order Flows and Aggregate Production

In this section we motivate the type of structure analyzed subsequently. We begin with some general observations about the consequences for aggregate economic activity of a large number of independent small fluctuations in final product demand. We argue that one needs both of the special features of our model – one the one hand, *local interaction* (each unit's production decision depends only upon the actions of a small number of other units that deal directly with it), and on the other hand, significant *nonlinearity* in producers' responses to demand variations – in order to obtain significant fluctuations in aggregate production.

In much macroeconomic analysis, the entire economy is modeled as a single market, in which the aggregate of all producers jointly supply goods to fulfill the aggregate demand of all consumers. It is assumed that only the joint productive capacity of all of the producers (described by an "aggregate production function") matters for describing the relation between final goods sales and producers' demands for primary inputs. Production is scheduled so as to minimize the costs of aggregate production. Because production possibilities exhibit diminishing returns to scale, the aggregate cost function is convex. (Specifically, we assume that both the direct cost of current production as a function of the flow rate of production, and the cost of carrying inventories as a function of the stock of inventories held, are increasing convex functions.) The consequence is that the cost- minimizing production plan makes production at each point in time a continuous function of the aggregate order flow. [8] Small variations in the aggregate order flow for final goods can then produce only small variations in aggregate production. But if the variations in final goods order are the aggregate of independent fluctuations in the demand of a large number of distinct final consumers, then there will be little variability in aggregate final goods orders. Furthermore, optimal production scheduling will make aggregate production even smoother than aggregate final sales, as inventory variations will be used to at least partially buffer variations in sales.

There are good reasons, however, to dispute the realism both of the assumption of a single aggregate market on the one hand, and of the assumption of

[8] One obtains, along standard lines, a continuous policy function from a concave discounted dynamic programming problem. For examples of the kind of theory of production scheduling in the face of stochastic final demand that we have in mind, see, e.g., Kollintzas (1989).

convex costs on the other. Yet we will argue that the picture is essentially the same, until *both* of the standard assumptions are modified.

It is obvious that actual economies are made of a large number of markets for distinct differentiated goods, especially when markets in different locations are treated as markets for distinct goods. It is likewise obvious that individual producers each buy from a small number of suppliers and sell to a small number of customers (if they are not producers of final consumer goods), where in each case we mean "small" compared to the total number of producers in the economy. The question is whether this pattern of local interaction matters for the analysis of aggregate fluctuations. In fact, introducing local interaction doesn't change our above conclusion materially, as long as we continue to assume convex costs.

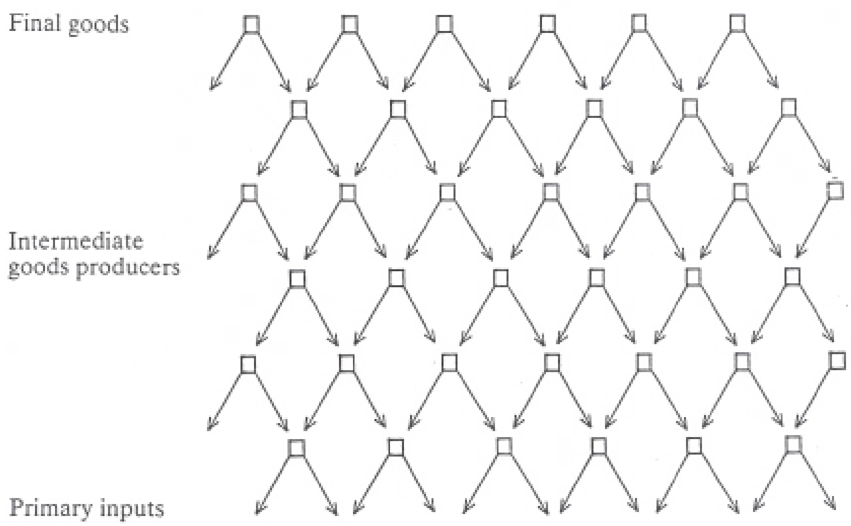

FIGURE A. Pattern of order flows.

Consider an economy made up of a large number of productive units, each of which supplies only a small number of customers, and in turn buys from only a small number of suppliers. For the sake of concreteness, we will suppose that the productive units are located on a cylindrical lattice like that shown in Figure A, with L rows and L units per row. Each unit buys supplies from the two units immediately below it, except the units in the bottom row, that purchase only primary inputs (that are produced without using any produced inputs, and the producers of which are therefore not represented on our lattice). Each unit correspondingly sells to the two firms immediately above it, except the units in the top row (final goods producers), that sell only to final consumers (who do not purchase goods as inputs for further production, and so are also not represented on the lattice). We wish to consider the fluctuations in production that occur in response to independent, stochastic fluctuations in the flow of purchases of the L distinct types of final goods.

Here we abstract from issues of pricing, for simplicity. That is, we assume fixed prices for each good, at which the supplier of that good stands ready to sell at all times. Because we assume that production is instantaneous once inputs are purchased, each unit is always able to immediately replenish its inventories if necessary to fill an order. It follows that all production units are constrained in the quantity that they can sell at any given time, but not in the quantity of inputs that they can purchase. Each unit's decision problem is then simply the scheduling of production (and the associated orders to its suppliers of inputs), given the random flow of orders that it receives. Each unit solves this problem so as to minimize the (discounted) sum of its direct costs of production and its costs of holding inventories.

The independent fluctuations in the demands for the different final goods are the sectoral "shocks" with which we will be concerned; they are small with respect to the aggregate if each final demand is an independent random variable with finite variance and L is large. We wish to consider a limiting case in which L becomes large, while the size of the random fluctuations in the flow demand for each final good remains the same. [9] We will consider what happens to the variability of aggregate production in the limiting case.

Let us first consider aggregate production by final goods producers. Since the fluctuations in the different producers' order flows are assumed to be independent, their production levels will fluctuate independently as well. Assuming similarly distributed bounded fluctuations in the production of each final good, aggregate production by final goods producers ceases to be variable as L is made large. [10]

Next consider aggregate production by producers whose immediate customers are final goods producers (*i.e.,* units in the second row of the lattice). Now distinct producers' order flows are not completely independent, as any given final goods producer will always simultaneously order inputs from two neighboring units in the second row. But each producer's order flow (and hence his production decision as well) will be independent of that of all producers

[9] Here we do not necessarily mean that the distribution of flow demands for each final good remains the same as L increases; in fact this is not true for the sequence of economies considered in the following section. But it is true in that example that the *bounds* upon the distribution of possible order flows to each final goods producer remains the same as L increases, and that the maximum possible fluctuation in orders to an individual unit eventually becomes an arbitrarily small fraction of the average value of aggregate final goods orders.

[10] For example, if the distribution of orders to each final goods producer remains the same as L increases, mean aggregate final goods production grows as L, while final goods production per sector is a random variable whose mean is independent of L and whose standard deviation falls as $L^{-1/2}$. In the example considered in the next section, mean aggregate final goods orders grow only as $L^{(1-\gamma)}$. In this case, aggregate final goods production scaled by $L^{(1-\gamma)}$ is a random variable whose mean is independent of L and whose standard deviation falls as $L^{-(1-\gamma)/2}$.

except the two units immediately adjoining it in the lattice. Thus the number of units with which any given unit is correlated becomes an arbitrarily small fraction of the total number as L is made large. Furthermore, the correlation between the variations in production by two adjoining units is less than perfect, and remains the same as L increases. Then a central limit theorem still applies in the case of limited dependence of this kind, and aggregate production by units in the second row also ceases to be variable as L becomes large. A similar argument applies for any given row i in the lattice.

Consider, instead, aggregate production by producers whose distance back in the production chain remains large compared to L, even as L increases. (After all, it is always the case that half the producers in the economy described are in rows $i > L/2$.) In this case, the demand faced by each intermediate goods producer depends (indirectly) upon the flow demand for final goods (currently and in the past) of many different sectors (all of its buyers' buyers' ... buyers) – in fact, of i different sectors. And a significant fraction of the other units in the same row will face demands that depend upon the demand for final goods in many of the same sectors. Hence a given unit's order flow (and hence its production decisions) may be significantly correlated with those of a significant fraction of other units in its row, even for arbitrarily large L. However, the order flows (and hence the production decisions) of these units should become less and less variable as L increases. If costs are convex, each unit's optimal current production will be a continuous function of its current order flow and current inventory. In the case of smooth costs and order fluctuations of only moderate size, the optimal response is well approximated by a *linear* function, so that current production can be written as a linear function of current and past orders from the unit's immediate customers. [11] A linear response function of this kind at each stage of production implies that each unit's production will depend only upon the *aggregate* final goods orders received by all of the unit's buyers' buyers' ... buyers, and upon past values of that aggregate. Then for large L this is the sum of a large number of bounded, independent random variables, and so a quantity with little variation relative to its average value. Furthermore, this reduction in the variability of production by each unit more than offsets the increased correlation between neighboring units (and the increased range of the correlation) as one goes back farther in the production chain. For assuming identical linear response functions for each unit, aggregate production in row i can be expressed as a linear function of current and past aggregate levels of final goods orders. The variability of aggregate final goods orders decreases with L, as discussed above. Furthermore, the farther back in the production chain one goes, the smaller the weight on current final goods orders and the longer the period of time over which past final goods orders are averaged in determining current production; hence aggregate production is actually less variable the farther back one goes. [12] It follows that as L increases, the variability of

[11] In the literature on production scheduling with convex costs, it is common to derive exactly linear response functions by assuming quadratic costs. See, *e.g.,* Kollintzas (1989).

[12] Not only does this model make it difficult to understand why sizeable aggregate fluctuations are observed; it also predicts, counterfactually, that the demand for final goods should

aggregate production (summing production by units at each of the stages of production) falls even faster than does the variability of aggregate final goods orders.

Another oversimplification in the standard macroeconomic model is the assumption of a convex production technology. Non-convex costs are in fact pervasive, due for example to indivisibilities – a significant part of the variation in plants' output occurs through starting and stopping of the operation of an entire assembly line or an entire shift. [13] The cost function associated with such indivisibilities may look something like the one shown in Figure B. [14] In such a case, average production costs are minimized by alternating between production at points A and B, given an average flow rate of sales less than B per period. Such non- convexities are now widely recognized to be important factors in production scheduling and inventory management at the plant level, where the empirical inadequacy of the "production smoothing" model described earlier has become evident. [15] More disputed is what significance, if any, such plant-level non-convexities have for macroeconomics.

be much more variable than the demand for intermediate goods, which is in turn more variable than the demand for primary inputs. Exactly the opposite appears to be observed, as is particularly evident from the greater cyclical variability of prices the farther back one goes in the production chain. See, *e.g.,* Murphy, Shleifer, and Vishny (1989). This is another puzzle that can be resolved by the model proposed here, although, as we do not consider endogenous price variation, the greater variability of demand for less finished goods shows up in our model as more variability in aggregate sales of such goods.

[13] See, *e.g.,* Davis and Haltiwanger (1990), Cooper and Haltiwanger (1992), or Bresnahan and Ramey (1992).

[14] Here we simply depict the direct costs of current production. Our remarks here about the consequences of indivisibilities for production scheduling do not depend critically upon the nature of the costs of carrying inventories, as long as these are not too large for inventories of the size associated with a production run of the size indicated by point B. In the model in the following section, this kind of non-convex function for direct production costs is combined with a standard convex function for the costs of holding inventories.

[15] See, *e.g.,* Blinder and Maccini (1991).

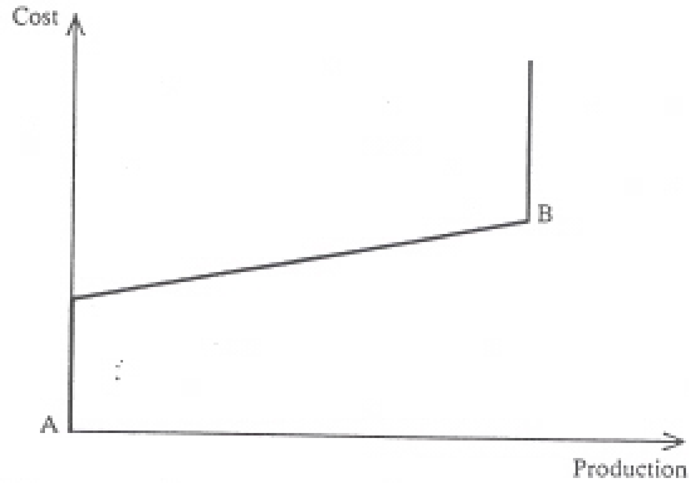

FIGURE B. The case of non-convex costs.

In fact, plant-level non-convexities also have little effect on our argument for the case of a simple aggregate model, if we continue to assume that there exists, in effect, a single market for a single good, and that the economy's total production possibilities are mobilized efficiently to meet aggregate demand. The non-convex costs in Figure B can obviously lead, in principle, to fluctuations in production that are larger than fluctuations in sales; one can even have fluctuating production in the case of a completely steady order flow (with periodic bursts of production to replenish inventories). But such a model can only explain aggregate fluctuations if the indivisibilities are large compared to the size of the whole economy. The standard argument for ignoring plant-level non-convexities is that even when each plant has a technology of this kind, aggregate production possibilities are well approximated by a convex cost function, if each plant is small compared to the scale of aggregate production. In the limiting case of a continuum of non-atomic plants (all producing the single good), the aggregate cost function is linear, regardless of the shape of the cost function for individual plants (assuming that increasing returns are exhausted for production by each plant above some efficient scale of operation). Hence one may argue that the effects of a fluctuation in aggregate sales should still be approximately as in the model with convex costs.

Thus neither local interaction nor non-convexity poses in itself a serious objection to the conventional result. We will show, however, that those two factors *in conjunction with one another* can yield very different results. Severely nonlinear local interactions, due to non-convexities at the level of the productive unit, result in non-additive aggregation of the effects of shocks to different sectors, with the result that the law of large numbers does not apply.

4 The Model

In our model, productive units are located on a cylindrical lattice, as shown in Figure A, and each has a production technology of the kind shown in Figure B. Each unit can be given coordinates (i, j), where $i, j = 1, 2, ..., L$. Here i is the row number, j is the column number, and we use modulo-L arithmetic for columns. Then unit (i, j) purchases goods from two suppliers, $(i+1, j)$ and $(i+1, j+1)$, if $i < L$; and sells goods to two customers, $(i-1, j)$ and $(i-1, j-1)$, if $i > 1$.

We assume a production technology in which average production costs are minimized by producing batches of two units of the good each time that production occurs (*i.e.,* point B in Figure B represents production of two units). We also assume that the expected time to arrival of the next order is small enough, relative to the rate at which future costs are discounted, so that even with discounting it is always optimal to produce two units at a time. Production of two units of output is assumed to require two units of inputs, one from each of the unit's two suppliers. (In the case of units in row L, we may also suppose that two units of primary inputs are required, though this is irrelevant for the production dynamics, since we assume that the primary inputs are always available when needed, and that purchases of them have no effect upon the demand for any of the produced goods.) Finally, we assume that the cost of holding one unit of inventory is negligible, while there is a significantly positive cost per period of holding more than one unit. Hence (given that inventories can always be replenished instantaneously by production) it is optimal for each unit to always hold in inventory either zero units or one unit of the good that it produces. New production only occurs when an order cannot be filled out of existing inventory, and never results in more than one unit remaining in inventory.

The initial state of the economy at the beginning of any period t is described by specification of the inventory holdings $x_{i,j}(t)$ for each productive unit (i, j). There are thus 2^{L^2} possible states in the configuration space X for this economy. Transitions between states occur in the following manner. Let $s_{i,j}(t)$ denote the number of sales by unit (i, j) in period t, and $y_{i,j}(t)$ the number of units of output produced by that same unit in the same period. Then inventory dynamics follow the law of motion

$$x_{i,j}(t+1) = x_{i,j}(t) + y_{i,j}(t) - s_{i,j}(t) \tag{1}$$

Furthermore, because of the considerations just mentioned, optimal production scheduling implies that output is a function of beginning-of-period inventories and the number of orders received,

$$y_{i,j}(t) = y(x_{i,j}(t), s_{i,j}(t)) \tag{2}$$

where the function $y(x, s)$ is defined in Table 1.

TABLE 1 *Laws of motion*

x	s	$y(x,s)$	$x'(x,s)$
1	0	0	1
1	1	0	0
1	2	2	1
0	0	0	0
0	1	2	1
0	2	2	0

Substitution of (2) into (1) implies that

$$x_{i,j}(t+1) = x'(x_{i,j}(t), s_{i,j}(t))$$

where the function $x'(x, s)$ is also defined in Table 1. In the Table, these functions are defined only for the values $x = 0, 1$ and $s = 0, 1, 2$. This is sufficient, because as long as each unit begins with either zero or one units of inventory and receives zero, one, or two orders, that unit will produce zero or two units, will end with zero or one units of inventory, and will order zero or one units from each of its suppliers, so that each its suppliers (which receives orders from only two customers) must also receive zero, one, or two orders.

The orders received by each unit with $i > 1$ are given by

$$s_{i,j}(t) = \frac{1}{2}(y_{i-1,j}(t) + y_{i-1,j-1}(t)) \tag{3}$$

The orders received by the units in the first row, $s_{1,j}(t)$, are specified as exogenous shocks, determined outside the system. Each of these is assumed to be either zero or one. The fact that a larger number of orders is not possible within a single period reflects an assumption that a "period" is very short; in particular, it is a time interval short enough for the discreteness of the units' order flows to be significant. Then the vector of exogenous shocks $s(t) = (s_{1,1}(t), ..., s_{1,L}(t))$ belongs to a shock space S of size 2^L. Equations (1) - (3) then completely determine the new state $x(t+1) \in X$ as a function of the initial state $x(t) \in X$ and the vector of exogenous shocks $s(t) \in S$.

The dynamics just specified are of the following character. Each of the random orders received by a final goods producer (unit with $i = 1$) initiates a chain reaction whose length depends upon the initial configuration. If the unit receiving the order can fill it out of existing inventory, no further orders are generated. But if it cannot, it produces and thus sends orders to two units in row $i = 2$. These firms may or may not then be required to produce, sending orders to firms in row $i = 3$. If they do, those orders may or may not trigger further production and further order flows, and so on. In the case that a final goods

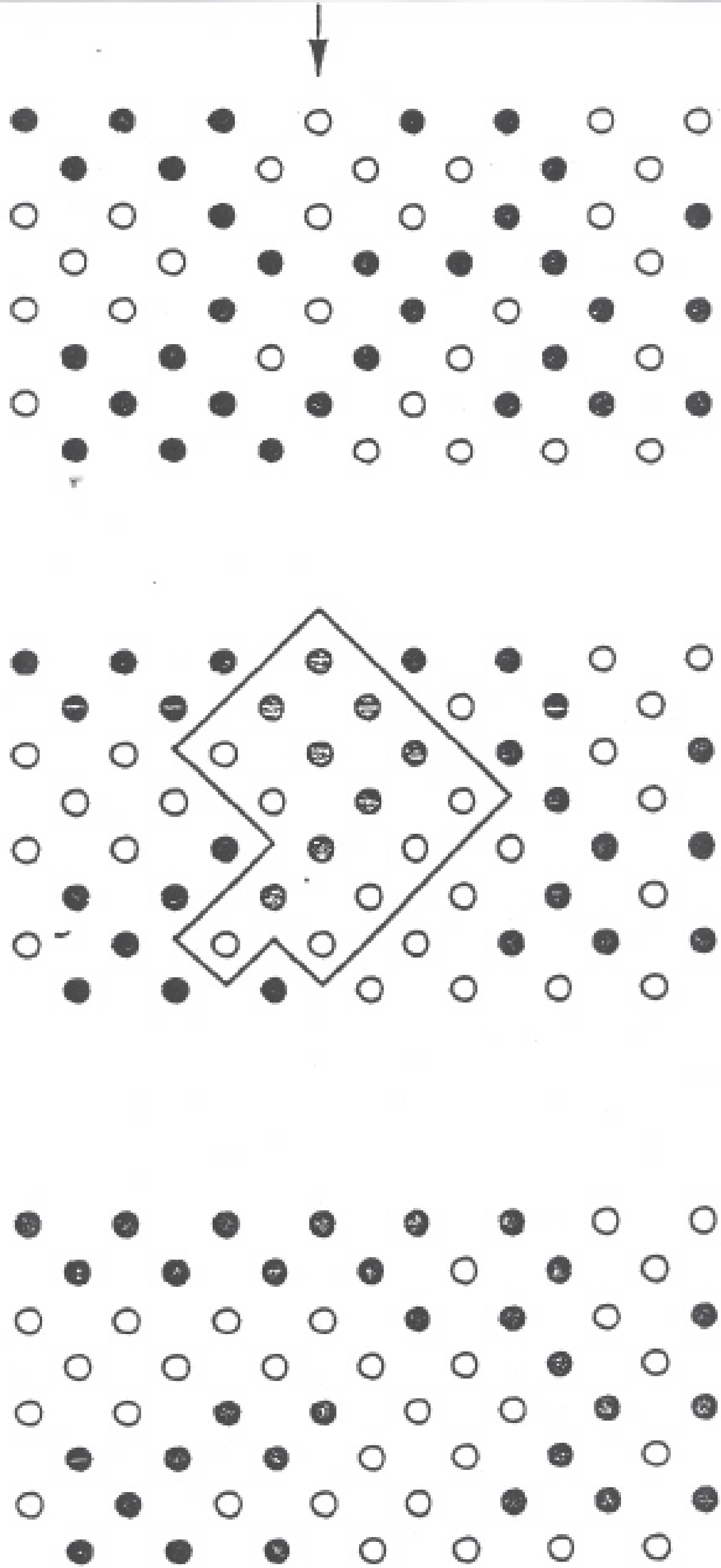

FIGURE C. The avalanche resulting from an initial final goods order at site indicated by the arrow.

order triggers production, we will refer to the resulting cascade of production by the final goods producer and its suppliers all the way back the production chain as an "avalanche". We will measure the "size" of such an avalanche by the total production that occurs by all of the firms involved.

As an example, consider the configuration shown in Figure C (top). Here black circles indicate units with an initial inventory $s_{ij} = 1$, and white circles units with initial inventory $s_{ij} = 0$. (Supply relationships are as in Figure A.) Suppose that an order is received by the final goods producer indicated by the arrow. As this producer has no inventory, it must produce in order to fill the order, leaving it with one unit of inventory after the order is filled. It orders supplies from two units in the row below, neither of which begin with any inventories. Each of these units also must produce in order to fill the orders, and each is left with one unit of inventory after the orders are filled. Each of these units orders supplies from two units in the row below. Thus three units in row three receive orders. The leftmost begins with one unit of inventory, and receives only one order, so it does not produce, but ends with zero inventory. The middle unit begins with zero inventory and receives two orders (one from each of the units that it supplies), so it produces and also ends with zero inventory. The rightmost of these units also begins with zero inventory, but receives only one order; it produces and ends with one unit of inventory. The unit that does not produce does not order supplies from its suppliers, but each of the other two units does, and so the process continues.

Figure C (middle) shows the size of the "avalanche" of production that eventually occurs as a result of the single final good order. The box encloses the set of productive units that are affected (that receive orders). Within this box, the grey circles indicate units that produce, while the white circles indicate units that simply fill an order out of existing inventory (and as a consequence end with zero inventory). Note that the lower boundary of the affected region consists entirely of white circles, while all units not on the lower boundary are grey circles – the avalanche stops if and only if orders can be filled out of existing inventories. In the case shown, the avalanche is of "size" 16; eight producers each produce two additional units of output. Figure C (bottom) shows the final configuration after all orders have been filled.

Given a probability distribution on the shock space S, and independent drawings of the shock vectors across periods, the dynamics above define a Markov chain on the configuration space X. We are interested in particular in the behavior of the following aggregate quantities.

$$N(t) = \Sigma_j s_{1,j}(t)$$

defines aggregate demand for final goods in period t, while

$$Y(t) = \Sigma_{i,j} y_{i,j}(t)$$

defines aggregate production. Note that both of these variables are functions of $(x(t), s(t))$. We are interested in whether it is possible for significant fluctuations in $Y(t)$ to occur despite an absence of significant exogenous fluctuations in aggregate final sales $N(t)$.

In the case of interest to us here, we assume that each of the exogenous random shocks $s_{1,j}(t)$, for $j = 1, ..., L$, is independent, taking the value zero with probability $1 - p$ and the value one with probability p, where p is a small positive number. These are the independent sectoral shocks whose aggregate effects we wish to analyze. Furthermore, we are interested in systems with a large number of such independent shocks. Thus we will consider systems in which the number of sectors L is made very large. In particular, we wish to consider the limiting behavior as L is made arbitrarily large, with p varying with the size of the economy, as some power $L^{-\gamma}$, with $2/3 < \gamma < 1$. [16] The mean number of final goods orders per period $N(t)$ is then $p(L)L$, which grows as $L^{1-\gamma}$. The random variable $\tilde{N}(t) = N(t)/L^{1-\gamma}$ then has a mean that does not change with L, and the limiting distribution of $\tilde{N}$, if it exists, is accordingly a reasonable indicator of the degree to which there are exogenous aggregate shocks in the large economy limit. It is easily seen that as L is made arbitrarily large, the random variable $\tilde{N}$ converges in distribution to a constant (the constant mean). Thus there exists no aggregate variability in the exogenous flow of final goods orders in the limit.

We now wish to consider a similar question about the limiting variability of aggregate production. It can be shown by arguments given in the following section that the median value of $Y(t)$ grows asymptotically as $L^{3(1-\gamma)}$. Hence we consider the limiting behavior of the scaled aggregate production measure $\tilde{Y}(t) = Y(t)/L^{3(1-\gamma)}$. [17] In the next section, we argue that $\tilde{Y}$ converges in distribution as L is made arbitarily large, and furthermore that the distribution is not a constant. In this sense we argue that aggregate fluctuations in production continue to occur in the large economy limit, even though aggregate exogenous shocks cease to exist.

5 The Distribution of Aggregate Production in the Large Economy Limit

Our study of the statistical properties of this model is simplified by observing that it is formally isomorphic to a sandpile model exhibiting self-organized criticality, that has been previously analyzed by Dhar and Ramaswamy (1989), who in turn exploit similarities between this type of model and a model of directed percolation analyzed by Domany and Kinzel (1984). [18] We first observe that for

[16] A possible interpretation of the variation of p with L is that as we consider larger systems we average our aggregate data over progressively shorter "periods".

[17] In this case we consider the median rather than the mean, because the appropriately scaled aggregate production turns out to converge to a distribution with no mean. In the case of final goods orders, the median also grows asymptotically as $L^{1-\gamma}$, but we refer above to the mean because its scaling properties are so trivial to derive.

[18] Dhar and Ramaswamy, however, are only concerned with the size distribution of the avalanches resulting from a single order for a final good. They thus do not consider the precise issue of interest here.

each possible vector of exogenous shocks $s \in S$, there is a well-defined transition operator $T_s : X \to X$. It is easily verified (i) that if $s_1 + s_2 = s$ (under modulo-2 arithmetic on each of the elements $s_{1,i}$), then $T_s = T_{s_1} \cdot T_{s_2} = T_{s_2} \cdot T_{s_1}$, and (ii) that each of the transformations is invertible (with inverse given by T_{-s}, where the $-s$ again refers to elementwise modulo-2 arithmetic). Thus the set of transformations $\{T_s\}$ for $s \in S$ forms an Abelian group, with T_0 the identity.

One consequence of this structure is that the effects of a shock vector s that involves orders at N sites are identical to the effects of a succession of N shocks, each involving an order at one of those sites. (Also, the order in which the individual final goods orders arrive does not matter.) Thus we may write

$$Y(t) = \sum_{j=1}^{N(t)} Y_j(t)$$

where $Y_j(t)$ represents the size of the avalanche caused by the order at the jth site at which an order is received in period t. Then our problem reduces to a study of the distribution of the sizes of the individual avalanches $\{Y_j(t)\}$.

Another immediate consequence of this structure is that the uniform distribution on the configuration space X is an invariant distribution for each of the transformations T_s, and hence is an invariant distribution for the Markov chain on X defined by our model. This allows us to calculate an unconditional probability distribution for Y_j. (Because of the symmetry of the model, this distribution is the same for all j, and is independent of the site at which the jth order is received.) For example, the unconditional probability that a final goods order triggers production by the unit receiving the order is exactly 1/2. Furthermore, conditional upon such production occurring (and orders being sent to two units in row 2), the probability that production is triggered at the two units in row 2 that supply the final goods producer is 1/2 in each case (and independent across the two producers). Thus the probability that an avalanche is of size zero is exactly 1/2 (the probability that $x_{1,i} = 1$ at the site where the initial order is received). The probability that an avalanche is of size two (only one unit produces) is exactly 1/8 (the probability that $x_{1,i} = 0, x_{2,i} = 1$, and $x_{2,i+1} = 1$). The rest of the probability distribution can be calculated in a similar manner.

Let R_j denote the last row affected by avalanche j, that is, the largest i such that $y_{i,k} = 2$ for some k. (We set $R_j = 0$ if no production occurs.) Then Dhar and Ramaswamy show that for all $r < L$,

$$P(R_j = r) = 2^{-2r-1}(2r)!/[r!(r+1)!]. \tag{4}$$

Note that $P(R_j = r)$ is independent of L, for all L large enough, so that there is a well-defined probability of reaching each row in the limiting case of an infinite lattice. In this limiting case, $P(R_j = r)$ declines as $r^{-3/2}$ for large r. Similarly, $P(Y_j = y)$ is independent of L for all L large enough (since $R_j \geq r$ necessarily implies $Y_j \geq 2r$.) Thus there is also a well-defined size distribution of avalanches in the case of an infinite lattice, and Dhar and Ramaswamy show that

$$P(Y_j > y) \sim y^{-1/3} \tag{5}$$

for large y. We will let π_L denote the unconditional distribution for Y_j in the case of a lattice of size L, and π_∞ the distribution in the limit of an infinite lattice. Numerical calculations of these distributions are plotted in Figure D. [19] We plot the logarithm of the frequency against the logarithm of the size of the avalanche, so that property (5) is evident in the asymptotic linearity of the plot of π_∞. [20]

This distribution π_∞ gives the probability of avalanches of various sizes, starting from an initial state randomly drawn using the uniform distribution. We next consider the long run frequency distribution of avalanche sizes when a system is observed over time. Unfortunately, the uniform distribution is not the only invariant distribution, and the system is not ergodic. The system posesses 2^{L-1} invariant classes. Each invariant class consists of all configurations such that $\sum_{j=1}^{L} x_{ij} = a_i$ modulo 2, for each $i > 1$, for some sequence of constants $a_i \in \{0, 1\}$. For each of these classes, the uniform distribution over the elements of the class is an extremal invariant distribution. Given any initial configuration, the long run frequency distribution of configurations visited is always one of these extremal distributions. However, the distribution of possible avalanche sizes, conditional upon the system being in a particular invariant class (specified by a sequence $\{a_i\}$) is independent of the class, and is the one derived above. Equation (4) is equally true when we condition upon the invariant class, for the probability of finding a zero or one at any site in the path of the avalanche (above row L) is the same, regardless of the value of a_i for that row. Similarly, $P(Y_j = y)$ for any y small enough compared to L is independent of the invariant class. Hence the limiting distribution of individual avalanche sizes, even conditioning upon the sequence $\{a_i\}$, is still the distribution π_∞ defined above.

It is obvious from (5) that π_∞ is a distribution with a very fat upper tail — in fact, it has no mean. Large avalanches are much more likely in this model than in the case of a Gaussian law (that would result if the size of an avalanche were the sum of a large number of independent, bounded random shocks). [21] It is because the long-run frequency distribution of avalanche sizes is π_∞ regardless of the initial state that we say that the system "self-organizes" to a state in which large avalanches are common. [22]

We now turn to a consideration of the unconditional distribution of the scaled aggregate production measure $\tilde{Y}$. Unfortunately, while the distribution π_L represents the unconditional distribution of Y_1, it is not the right conditional

[19] See the next section for discussion of the numerical calculations.

[20] Note that (5) implies that the frequency should decline as $y^{-4/3}$, which is the slope observed in the plot.

[21] The power-law relation between the size of the event and its frequency displayed in (5) has been much discussed as an indication of self-organized criticality (*e.g.,* Bak, Tang, and Wiesenfeld (1988)).

[22] In the sandpile model discussed in section 2, a sandpile that is relatively flat corresponds to an initial configuration in which most of the $x_{i,j}$ are ones. In such a case almost all avalanches are small. But as indicated before, such a state will not last as grains of sand are added at random; formally this is proved by showing that states of this kind are given little probability weight in the long-run frequency distribution on X, regardless of the invariant class to which the initial configuration belongs.

distribution for Y_2 given Y_1, for the sizes of successive avalanches are not independent random variables. In particular, a large avalanche starting at one site makes it harder for a large avalanche to occur starting at a nearby site. Recall the region affected by the avalanche in Figure C (middle). One observes that in the final configuration, every unit on the upper boundary of the affected region (those units belonging to the region but with a customer that does not) that is not also part of the lower boundary (those units belonging to the region but with a supplier that does not) has an inventory of one unit. Thus the upper boundary of the region affected by one avalanche becomes a wall of units holding inventory, so that a subsequent avalanche beginning outside the region is not able to penetrate it (as these units can each fill an order with out passing on any orders to their own suppliers). The region affected by the second avalanche can overlap with the first region only at boundary units of the first region. Thus if the first affected region is large, and the second avalanche starts nearby, it is hard for the second avalanche to affect a large region.

This dependence complicates our analysis. However, we argue that its effects become negligible in the limit of a large system, if the probability $p(L)$ falls with L at a fast enough rate. Our argument proceeds in several steps.

(1) First we consider the probability distribution $\bar{F}_{\infty}^{N}$ of a variable $W_N = \sum_{j=1}^{N}(Y_j/N^3)$, for given N, if the Y_j are independent drawings from π_∞, the unconditional distribution for the size of individual avalanches in an infinite lattice. Property (5) and well-known results on the domain of attraction of stable laws (see,*e.g.,* Gnedenko and Kolmogorov (1968), p. 175, Theorem 2) imply that as $N \to \infty$, this distribution converges to a certain distribution that we will denote F_{∞}^{∞}, a Pareto-Levy stable law with exponent $\alpha = 1/3$. The logarithm of the characteristic function of this distribution is given by

$$g(t) = i\delta t - c|t|^{1/3}\big[1 - i(t/|t|)tg(\pi/6)\big] \tag{6}$$

where $c = \int_0^\infty [(1-e^{-x})/x^{4/3}]dx\ cos(\pi/6)$. The probability density for F_{∞}^{∞} is plotted in Figure E(iv) (solid line). [23] Here we again plot the logarithm of the density against the logarithm of y; the asymptotic linearity of this plot again indicates that the probability density falls as $y^{-4/3}$ for large values of $y = lim_{N\to\infty}W_N$. Thus the size distribution of this aggregate inherits the property (5) of the size distribution of individual avalanches, and again, large aggregate fluctuations are much more frequent than would occur in the case of a normal law.

(2) We next consider the probability distribution F_{∞}^{L} of the variable $Z_L = \sum_{j=1}^{N_L}(Y_j/N_L^3)$, where N_L is a random variable, and again the $\{Y_j\}$ are independent drawings from π_∞. Here N_L is the random value of $N(t)$, the number of total final goods orders, if the number of distinct goods is L and the probability of each order is $p(L) = kL^{-\gamma}$ for some constant $k > 0$. We assume that N_L is independent of each of the $\{Y_j\}$. We now show that as $L \to \infty$, fixing k and γ, $F_{\infty}^{L} \to F_{\infty}^{\infty}$ as well (where F_{∞}^{∞} is again the distribution defined by (6)).

[23] The numerical method used to compute it is discussed further in the next section.

Fix $x > 0$ a point of continuity of F_∞^∞. For any $\epsilon > 0$, let L be large enough so that

$$P\{(k/2)L^{1-\gamma} < N_L\} > 1 - \epsilon/3. \tag{7}$$

Note that if $\gamma > 0$, $E(N_L) \sim L^{1-\gamma}$, and $Var(N_L) \sim L^{1-\gamma}$ for large L. Then Chebyshev's inequality implies that if $0 < \gamma < 1$, (7) must hold for all L large enough. It follows that

$$|F_\infty^\infty(x) - F_\infty^L(x)| = |F_\infty^\infty(x) - P\{W_{N_L} \leq x\}|$$

$$= |F_\infty^\infty(x) - P\{W_{N_L} \leq x \textit{ and } (k/2)L^{1-\gamma} < N_L\}$$

$$-P\{W_{N_L} \leq x \textit{ and } (k/2)L^{1-\gamma} \geq N_L\}|$$

$$\leq |F_\infty^\infty(x) - P\{W_{N_L} \leq x \big| (k/2)L^{1-\gamma} < N_L\} \cdot P\{(k/2)L^{1-\gamma} < N_L\}| + \epsilon/3 \tag{8}$$

where $P(\cdot\big|\cdot)$ denotes the conditional probability.

Furthermore, since $\bar{F}_\infty^N \to F_\infty^\infty$ as $N \to \infty$, we know that if L is large enough, and $N > (k/2)L^{1-\gamma}$,

$$|F_\infty^\infty(x) - P\{W_N \leq x\}| < \epsilon/3.$$

Therefore

$$|F_\infty^\infty(x) - P\{W_{N_L} \leq x \big| (k/2)L^{1-\gamma} < N_L\}| < \epsilon/3.$$

Then (7) together with $0 \leq F_\infty^\infty(x) \leq 1$ imply that

$$|F(x) - P\{W_{N_L} \leq x \big| (k/2)L^{1-\gamma} < N_L\} \cdot P\{(k/2)L^{1-\gamma} < N_L\}| < 2\epsilon/3.$$

This together with (8) then implies that

$$|F_\infty^\infty(x) - F_\infty^L(x)| < \epsilon.$$

Thus $F_\infty^L(x) \to F_\infty^\infty(x)$ as $L \to \infty$, for x any point of continuity of $F_\infty^\infty(x)$.

(3) We next consider the probability distribution F_L^L of the variable $V_L = \sum_{j=1}^{N_L}(Y_{j,L}/N_L^3)$, where now the $\{Y_{j,L}\}$ are independent drawings from π_L, and N_L is an independent random variable with the same distribution as above. We present a heuristic argument that $F_L^L \to F_\infty^\infty(x)$ as $L \to \infty$. Note that $V_L(\omega) = Z_L(\omega)$ for each state ω in which each of the N_L avalanches in the infinite lattice happens to die out before row L of the lattice is reached. [24] The probability of this occurring is $\psi_L = \sum_{n=1}^\infty P(N_L = n) \cdot (1 - \phi_L)^n$, where $\phi_L = P(R_j \geq L)$ for an individual avalanche j. For large L, this quantity is approximately $(1 - \phi_L)^{kL^{1-\gamma}}$. Recall furthermore that by (4),

$$\phi_L \sim L^{-1/2}$$

[24] Here a typical element ω of the underlying probability space represents a drawing of N_L and of N_L drawings of configurations for the infinite lattice. Even though the uniform invariant distribution for configuration space X is improper in the case of an infinite lattice, the probabilities of all cylinders of the form "$N_L = 2$, with the first configuration having $x_{1,1} = 1$, and the second configuration having $x_{1,1} = 0, x_{2,1} = 1$, and $x_{2,2} = 1$" are well-defined.

for large L, so that

$$-log(1-\phi_L) \sim L^{-1/2}$$

$$-log(\psi_L) \sim L^{1/2-\gamma}$$

which latter quantity converges to zero as $L \to \infty$, provided that $\gamma > 1/2$. Hence if $\gamma > 1/2$, $\psi_L \to 1$ as $L \to \infty$. But then for any $x > 0$,

$$|F^L_\infty(x) - F^L_L(x)| \leq 1 - \psi_L$$

so that $|F^L_\infty(x) - F^L_L(x)| \to 0$ as $L \to \infty$. This implies that $|F^\infty_\infty(x) - F^L_L(x)| \to 0$ as $L \to \infty$ as well.

(4) We next consider the unconditional probability distribution G_L of $\bar{Y}_L = Y_L/N^3_L$, where Y_L and N_L are the aggregates for our model defined in the previous section (aggregate production and aggregate final goods sales, respectively), in the case of a lattice of size L. Note that $\bar{Y}_L$ is the same random variable as V_L, except that in our model the sizes of successive avalanches are not independently distributed (as explained above). We now provide a heuristic argument that the dependence should become negligible for large L, so that $G_L \to F^\infty_\infty$ as $L \to \infty$, just as in the case of F^L_L.

We consider the probability of an initial configuration, and a vector of final goods sales, such that, however many avalanches occur, none of the affected regions overlap, and furthermore the affected regions for any two neighboring avalanches are separated by at least one site in each row. Conditional upon this event E, the size distributions of all of the avalanches are independent,and are furthermore the same regardless of which invariant class to which the initial configuration belongs. (If the affected regions do not jointly exhaust any row, then the invariant distribution of possible states at sites in the paths of the avalanches is the uniform distribution, independent of the invariant class. And if the affected regions do not overlap, the initial states at sites in the path of one avalanche are independent of the initial states in the path of any other avalanche.) Hence the distribution of $\bar{Y}_L$ is identical to that of V_L. Thus it suffices to show that the probability of event E converges to 1 as $L \to \infty$.

In a lattice of size L, consider an avalanche starting at site $(1, j)$, and suppose that the next site to the right at which an avalanche starts is $(1, j + q + 1)$, for some $q > 0$. Let $v(i)$ denote the right boundary of the affected region of the first avalanche in row i (defined for all $i \leq R_1$, where R_1 is the last row reached by that avalanche), and let $u(i)$ denote analogously the left boundary of the affected region of the second avalanche. We wish to bound below the probability that $v(i) + 1 \leq u(i)$, for all i such that both are defined. Note that conditional upon the fact that neither of the avalanches terminates at or before row i, and that $v(i)+1 \leq u(i)$, $v(i+1) = v(i)$ with probability $1/2$ and equals $v(i)+1$ otherwise, and u is an independent random walk with drift, of the same kind.

The probability that $v(i) + 1 \leq u(i)$ for all relevant i is in turn bounded below by the probability that

$$v(i) < j + (\frac{i+q-1}{2}) \tag{9a}$$

$$u(i) > j + (\frac{i+q+1}{2}) \qquad (9b)$$

both hold for all relevant i. In order to obtain an approximation of this probability, we consider the limiting case of a continuous lattice and a Brownian motion with drift. The probability that a Brownian motion with drift 1/2 per row (where "row" is now a continuous time-like coordinate) and instantaneous variance 1/4 per row never reaches a distance $q/2$ to the right of its expected path, over the entire interval $[0, L]$, is just the probability that M_L, the maximum over $[0, L]$ of a Brownian motion with this variance but no drift, is less than $q/2$. But it is well-known that M_L is distributed as $|B_L|$, the absolute value of the latter Brownian motion at row L. In turn,

$$P\{|B_L| \geq q/2\} \leq L/q^2,$$

by Chebyshev's inequality (since the variance of B_L is $L/4$). Hence the probability that (9a) holds for all $i \leq L$ is at least $1 - L/q^2$. Thus the probability that both (9a) and (9b) hold is at least $(1-L/q^2)^2$, and likewise the probability that the two affected regions are always separated by at least one site.

In the case of N avalanches, each starting at a site a distance $q + 1$ from the previous one, one similarly shows that the probability that none of the N affected regions adjoin is at least $(1 - L/q^2)^{2N}$. Now if $p(L) = kL^{-\gamma}$, the mean distance between sites at which final goods orders are received in a given period is $q + 1 = k^{-1}L^{\gamma}$, and the mean number of such sites is $N = kL^{1-\gamma}$. The probability that none of the regions adjoin is then bounded below by

$$\left(1 - k^2 L^{1-2\gamma}\right)^{2kL^{1-\gamma}}$$

If $1/2 < \gamma < 1$, the logarithm of this probability is of the order of $-L^{2-3\gamma}$ for large L, so that if $2/3 < \gamma < 1$, the probability of event E converges to one as $L \to \infty$. Hence we expect G_L and F_L^L to approach the same limit as $L \to \infty$, which is to say that $G_L \to F_\infty^\infty$.

(5) Finally, we consider the unconditional probability distribution H_L of $\tilde{Y}_L/k^3 = Y_L/k^3 L^{3(1-\gamma)}$, where Y_L is the same aggregate as above. Now $\tilde{Y}_L/k^3 = \bar{Y}_L \cdot (N_L^3/k^3 L^{3(1-\gamma)}) = \bar{Y}_L \cdot (\tilde{N}_L^3/k^3)$, and $\tilde{N}_L^3/k^3 \to 1$ (a constant) with probability one as $L \to \infty$. Then by Slutsky's theorem, $H_L \to F_\infty^\infty$, the same limit as G_L, as $L \to \infty$.

Thus we can establish that in the large economy limit, the distribution of values taken by the scaled aggregate output $\tilde{Y}$ converges to a non-degenerate probability distribution, although the scaled level of aggregate final goods sales $\tilde{N}$ converges to a constant. [25] Furthermore, the limiting distribution is one that allows levels of aggregate output that are large compared to the median to occur with great frequency — the distribution has no mean, and the upper tail follows a power law of the kind discussed above in connection with the size distribution of individual avalanches.

[25] This result also establishes that the median of Y_L, when scaled by $L^{3(1-\gamma)}$, approaches a constant, namely $k^3 med(F_\infty^\infty)$, as asserted at the end of the previous section.

6 Simulations of a Finite Economy

In this section we present the results of numerical simulations of the model described in section 4. These results help to give some idea of how accurate the conclusions of the previous section regarding behavior in the large economy limit are as a description of what happens in a finite economy.

First we consider the unconditional probability distribution for the size of an individual avalanche. Calculation of this does not require simulation of the model of section 4. It suffices to simulate the evolution of the left and right boundaries of the affected region, the laws of motion of which are described in the previous section. An avalanche terminates when the right boundary crosses the left boundary.

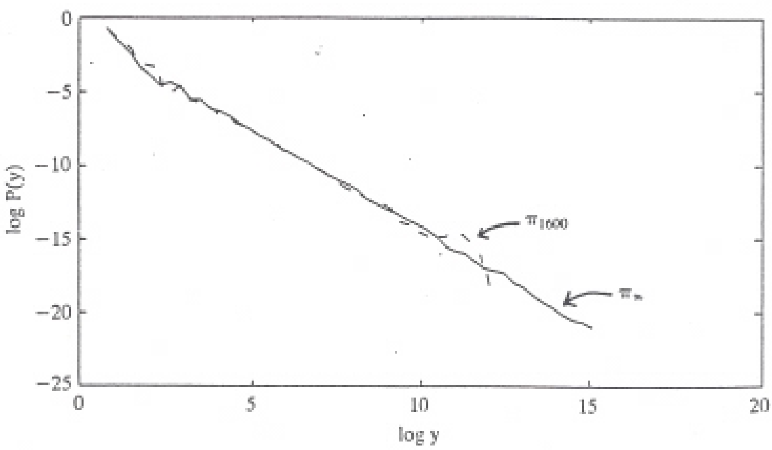

FIGURE D. Distributions of individual avalanche sizes.

Figure D shows our results for the case of avalanches truncated at row L = 1600, and for avalanches in an infinite lattice. As our previous theoretical argument indicated, the distributions π_{1600} and π_∞ coincide for avalanches up to a certain size; here we see that they are similar for avalanche sizes up to around 15,000. On the other hand, the frequency of avalanches of size greater than 50,000 falls off much faster in the case of the finite lattice than in the case of the infinite lattice (where frequency declines with size only at the rate indicated by (5)). Avalanches of size 30,000 or so are actually more frequent in the system of size 1600 than in the infinite system; this is because many avalanches that would continue if not truncated at row 1600 are of roughly this size.

Figure E(i) (solid line) plots the density function for the distribution H_{1600}, the distribution of the scaled aggregate output measure $\tilde{Y}/k^3$, for an economy with $L = 1600$ and $p = .005$. Over a certain range (values between 0.2 and

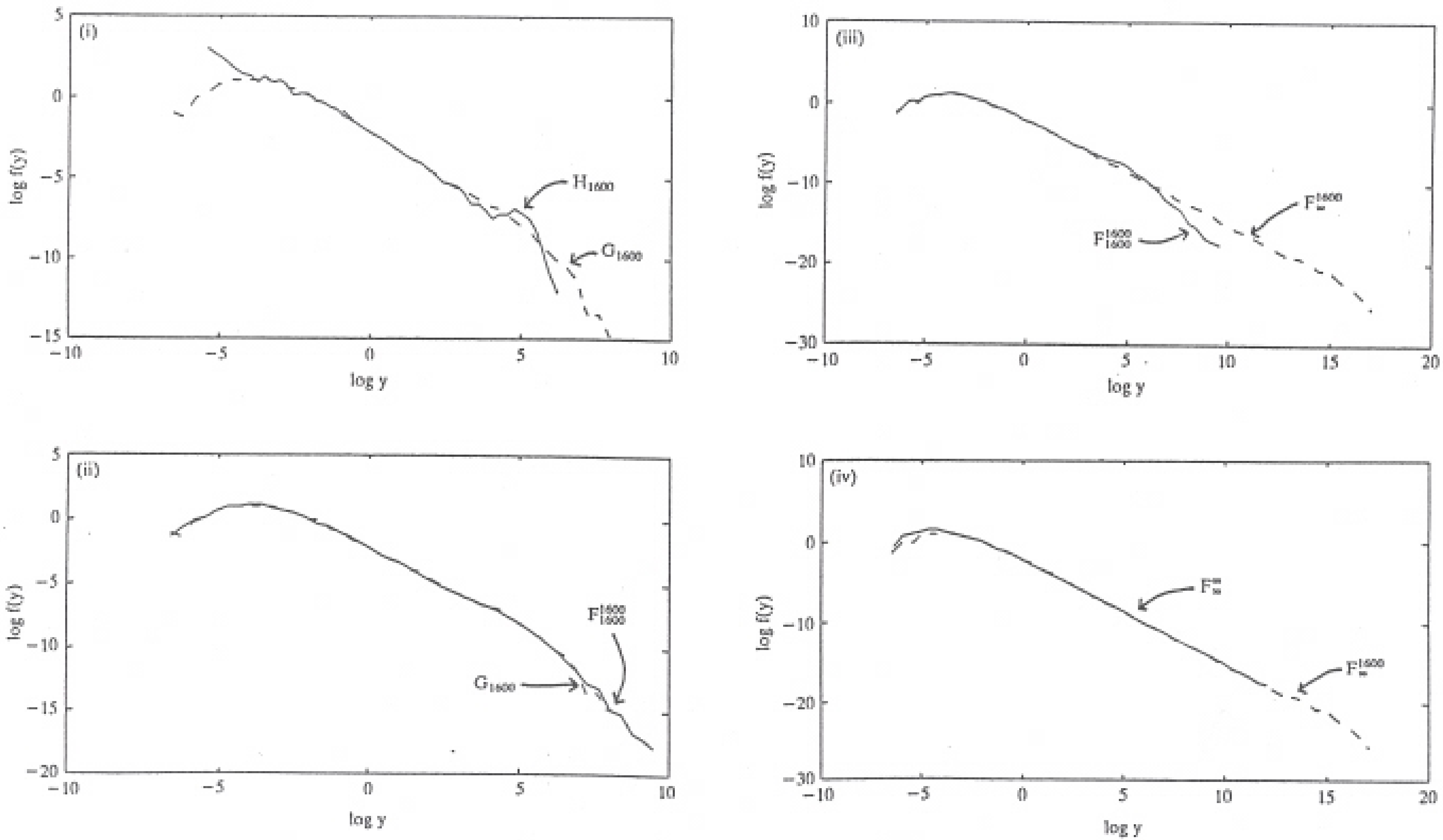

FIGURE E. Distributions of aggregate activity.

20) the logarithm of the density varies roughly linearly with aggregate output. However, this is certainly not true for larger output fluctuations, whereas our theory (for the large economy limit) predicts *asymptotic* linearity, for the largest values. The remaining distributions plotted in Figure E are intended to give insight into the exact ways in which the large economy limit fails to be a good approximation for this model.

Figure E(i) compares H_{1600} with G_{1600}, the distribution of Y/N^3 for our model. We know that for large L these distributions are the same, but even for $L = 1600$ we see that they are quite different at the extremes. This is because the binomial distribution for the number of final goods sales N is not yet too well approximated by a constant; the mean is 8, but the standard deviation is nearly 3, which means that variations that are large relative to the mean occur relatively frequently. This problem eventually disappears, however, as L is increased, if p varies with L in the way assumed above. [26]

Figure E(ii) compares G_{1600} with F_{1600}^{1600}, the distribution of Y/N^3 if the sizes of successive avalanches are independent draws from the unconditional distribution. Note that these distributions are indistinguishable, at the level of resolution allowed by our numerical work. Thus the dependence that exists in our model between the sizes of successive avalanches is not quantitatively significant, except perhaps for avalanches so large that they do not occur often in a data set of the size used in preparing the Figure. [27]

Figure E(iii) compares F_{1600}^{1600} with F_{∞}^{1600}, the distribution of Y/N^3 if the avalanches are not only independent but occur in an infinite lattice (though the distribution of the number of avalanches is the same as in our model with $L = 1600$ and $p = .005$). Here we see that truncating the avalanches at row 1600 has a similar effect on the distribution of this aggregate measure as it does on the size distribution of individual avalanches (shown in Figure D). The distributions coincide for small values of y (up to about 20), the distribution for the finite lattice makes values of a certain size (about 100) more frequent than it would be in the case of an infinite lattice, and beyond that scale the density falls off more rapidly with size in the case of the finite lattice. Notice also that the distribution for avalanches in an infinite lattice exhibits a roughly linear graph (the power-law behavior discussed above) over a large range of values of y from about .05 to the highest values observed (over 10 million). (Deviations from exact linearity in our plot are probably mainly due to not having a large enough sample to estimate the upper tail very accurately.)

Finally, Figure E(iv) compares F_{∞}^{1600} with F_{∞}^{∞}, the theoretical large-economy limit. (Here we approximate the limit not by making L large and p small, but by computing $\bar{F}_{\infty}^{N}$ for large N. [28]) Here there is little visible difference between

[26]The problem with our present example is not so much that L is small as that p is quite small (our "period" is short). We have chosen a small p in order to make interference between avalanches at different locations within a single "period" infrequent, as the asymptotic absence of such interference plays a crucial role in our argument in the previous section.

[27]We simulated the model for 5000 periods in the case of G_{1600}, and simulated 60,000 independent avalanches in the case of F_{1600}^{1600}.

[28]For the Figure, we use $N = 20$, since variation in N already has little effect on the

F_∞^{1600} and the limiting distribution. To sum up, the observed differences between H_{1600} and the Pareto-Levy distribution predicted for the large-economy limit are almost entirely due to (i) the fact that in our example, aggregate final goods sales are still not at all constant, and (ii) the fact that individual avalanches truncated at row 1600 have a different distribution (at the upper tail) than do avalanches in an infinite lattice.

computed distribution.

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