Seminar on solid geometry

January 24, 2024

The objective of this seminar is to give an introduction to the new theory of analytic stacks of Clausen and Scholze, taking as main examples analytic stacks arising from rigid and adic spaces. The main reference for the course will be the lectures on analytic stacks (AS) that can be found in `https://people.mpim-bonn.mpg.de/scholze/AnalyticStacks.html`. We will also use other auxiliary references such as the manuscripts of previous condensed courses [CS19, CS20, CS22], Mann's thesis on six functors [Man22b, Man22a], the manuscript of six functors [Sch23], and the paper [RC24].

Following the line of the course of analytic stacks, we shall review the theory of light condensed sets and light condensed abelian groups, we introduce the category of light solid abelian groups, we review the theory of analytic rings in condensed mathematics making emphasis in the theory of solid analytic rings. In a second part of the seminar, we review the theory of abstract six functor formalism and explain how it is a key tool in the definition of analytic stack. Finally, we will focus on examples arising from classical algebraic, rigid and adic geometry.

1 Talks

1.1 Introductory talk (Lecture 1 of AS)

1.2 Light condensed sets and abelian groups (Lectures 2-4 of AS, [CS19, I-II])

We define light condensed sets and the category of light condensed abelian groups. We relate classical topological spaces with light condensed sets ([CS19, Proposition 1.7]), in particular we show that in good cases (metrizable locally compact Hausdorff spaces) condensed cohomology also computes sheaf cohomology ([CS19, Theorems 3.2 and 3.3]). Finally, we prove that the condensed abelian group of null sequences is internally projective.

1.3 Light solid abelian groups I (Lecture 5 of AS, [CS19, IV-VI])

We introduce the category of light solid abelian groups , we describe its compact projective generators as well as the solid tensor product (cf. [CS19, Theorem 5.8 and Proposition 6.3]).

1.4 Light solid abelian groups II (Lecture 6 of AS, [CS19, IV-VI])

We describe the finitely presented solid modules and prove flatness of $\prod_{\mathbb{N}} \mathbb{Z}$. Finally, we give some examples of solid tensor products appearing in rigid or formal geometry (cf. [RJRC22, Lemmas 3.13 and 3.28]).

1.5 Analytic rings I (Lecture 13 of AS, [CS20, XI-XII])

We define the notion of analytic ring ([CS20, Definition 12.1] and [Man22b, Definition 2.3.1]) and prove some basic categorical properties: completion of analytic rings ([CS20, Proposition 12.26] and [Man22b, Lemma 2.3.11]), the existence of colimits of analytic rings ([CS20, Proposition 12.12] and [Man22b, Proposition 2.3.15]), and the formation of the induced analytic ring structure ([Man22b, Definition 2.3.13]).

1.6 Analytic rings II (Lectures 7 and 13 of AS, [CS20, Lecture XII: Appendix]), [CS19, VIII]

We prove the invariance of an analytic ring structure under π_0 ([CS20, Proposition 12.21]) and nilpotent thickenings ([CS20, Proposition 12.23]). Then, we discuss a very general way to construct analytic rings by localizing with respect to suitable modules. As particular case we recover the ring of solid integers $\mathbb{Z}_{\blacksquare}$. We then give the examples of the solid affine line $\mathbb{Z}[T]_{\blacksquare}$ ([CS19, Theorem 8.1]) and the ultrasolid rational numbers $\mathbb{Q}_{\blacksquare}$.

1.7 Solid analytic rings I (Lectures 8 and 11 of AS), [CS19, Lecture VIII]

We study the analytic ring $A_{\blacksquare}$ where A is an algebra of finite type over $\mathbb{Z}$: we compute the compact projective generators and prove flatness of $\prod_{\mathbb{N}} A$. Finally, we define generalized affinoid rings $(A, B)_{\blacksquare}$.

1.8 Solid analytic rings II (Lectures 9 and 10 of AS), [CS19, Lecture IX-X]

We see how the formalism of analytic rings gives a perfect framework to study discrete Huber rings and Tate affinoid rings (A, A^+). In particular, we prove that solid quasi-coherent sheaves satisfy descent for the analytic topology of the underlying adic space ([CS19, Theorem 9.8] and [And21, Theorem 4.1]).

1.9 Six functor formalisms (Lecture 15-18 of AS, [Man22a, §5-9], [Sch23, II-V], [RC24, §3.1])

We give a brief introduction to abstract six functor formalisms: we explaim how six functor formalisms are constructed from a class of "proper" and "étale" maps ([Man22b, Proposition A.5.10] and [Sch23, IV]). We define closed and open immersion of symmetric monoidal categories, motivated from six functors ([CS22, V]). We define smooth and descendable morphisms for abstract six functor formalisms and show that they satisfy $*$ and !-descent ([Sch23, V, Appendix VI]).

1.10 Six functors for analytic rings (Lectures 15-18 of AS, [RC24, §3.2])

We define the six functor formalisms for analytic rings by taking defining closed and open morphisms of analytic rings. We mention Clausen and Scholze's proof that !-descent is equivalent to universal $*$ and !-descent, and also equivalent to descent of modules in presentably symmetric monoidal categories.

1.11 Analytic stacks (Lecture 19 of AS)

We give the formal definition of analytic stacks using the !-topology, then we give examples of analytic stacks: light condensed anima, derived schemes, rigid spaces, complex and real analytic spaces, etc.

1.12 Rigid spaces as analytic stacks and GAGA (Lectures 15 and 19 of AS)

We show that rigid spaces can be constructed as analytic stacks. We define an analytification functor from schemes to rigid spaces, and show a geometric GAGA statement ([CS22, VI-VII]).

1.13 Smooth morphisms of rigid spaces ([RC24, §3.5])

We review the definition of smooth and étale morphisms of rigid spaces using the theory of analytic stacks. We define cotangent complexes of analytic rings and show that a morphism of smooth rigid spaces is smooth if and only if it is formally smooth and of (solid) finite presentation ([RC24, Theorem 3.5.6]).

1.14 Serre duality ([CS19, XI], [CS22, XIII], [RC24, §3.6])

We prove Serre duality for smooth morphisms of rigid spaces ([CS22, Theorem 13.6] and [RC24, Theorem 3.6.15]).

References

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[CS19] Dustin Clausen and Peter Scholze. Lectures on Condensed Mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.

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[RC24] Juan Esteban Rodríguez Camargo. The analytic de Rham stack in rigid geometry. `https://arxiv.org/abs/2401.07738`, 2024.

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