1D H-box Method for Shallow Water Equations

with zero-width barrier

Chanyang Judah Ryoo

APAM
Columbia University

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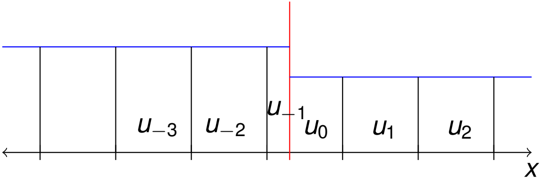
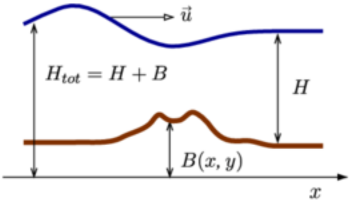
Why Zero-width barrier?

And the difficulties.



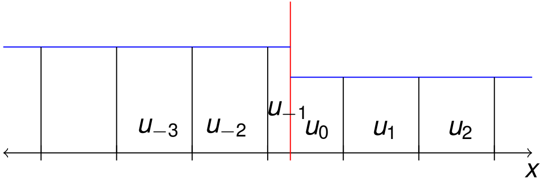
Shallow Water Equations



$$h_t + (hu)_x = 0$$
$$(hu)_t + (\frac{1}{2}gh^2 + hu^2)_x = -\frac{1}{2}ghb_x \quad (1)$$

Why Zero-width barrier?

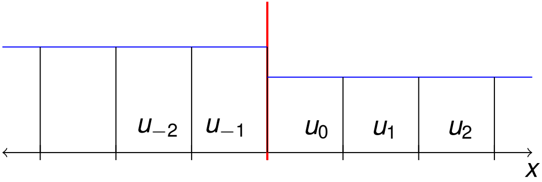
And the difficulties.



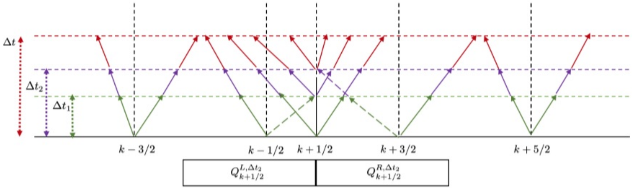
- Small cells: $\alpha\Delta x$, $(1-\alpha)\Delta x$
- No water on top of wall
- Flux

Previous Work (J. Li 2019)

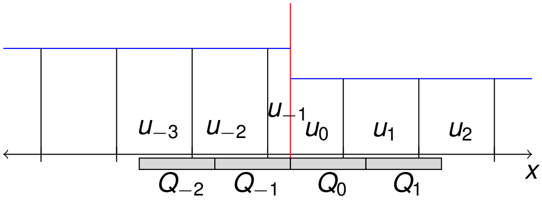
- Wall on an edge



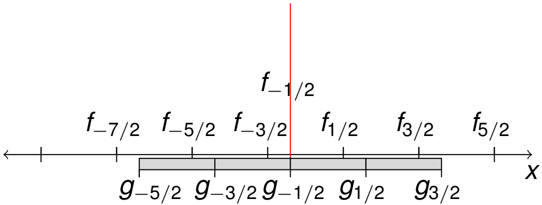
- "Large-time-step" method for wall off edge



The Idea: H-Box Method



The Idea: H-Box Method

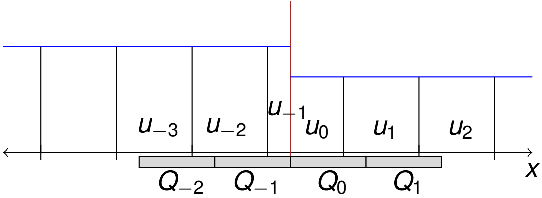


$$g_{-1/2} := f_{-1/2} \tag{2}$$

$$g_{1/2} := f_{1/2}, \quad g_{3/2} := \alpha f_{5/2} + (1-\alpha) f_{3/2} \tag{3}$$

$$g_{-3/2} := f_{-3/2}, \quad g_{-5/2} := \alpha f_{-5/2} + (1-\alpha) f_{-7/2} \tag{4}$$

The Idea: H-Box Method

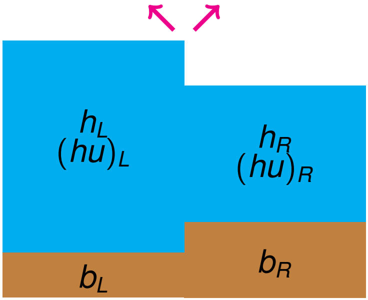


$$u_0^{n+1} := Q_0^{n+1}, \quad u_{-1}^{n+1} := Q_{-1}^{n+1} \tag{5}$$

$$u_1^{n+1} := \alpha Q_0^{n+1} + (1-\alpha) Q_1^{n+1}, \quad u_2^{n+1} := \alpha Q_1^{n+1} + u_2^{n+1} \tag{6}$$

$$u_{-2}^{n+1} := \alpha Q_{-2}^{n+1} + (1-\alpha) Q_{-1}^{n+1}, \quad u_{-3}^{n+1} := (1-\alpha) Q_{-2}^{n+1} + u_{-3}^{n+1} \tag{7}$$

Riemann Problem



Riemann Problem Solver (D. George 2008)

$$\begin{bmatrix} h_R \\ (hu)_R \\ \phi_R \end{bmatrix} - \begin{bmatrix} h_L \\ (hu)_L \\ \phi_L \end{bmatrix} - \Psi(q_L, q_R) = \begin{bmatrix} 1 & 0 & 1 \\ s_\epsilon^1 & 0 & s_\epsilon^2 \\ (s_\epsilon^1)^2 & 1 & (s_\epsilon^2)^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix},$$

where $\phi = \frac{1}{2}gh^2 + hu^2$,
$\Psi(q_L, q_R)$ = source term arising from bathymetric variation, and
$s_\epsilon^{1,2}$ = two eigenvalues arising from system of SWE, 'speeds'

Riemann Problem Solver (D. George 2008)

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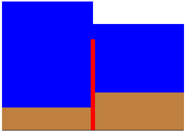
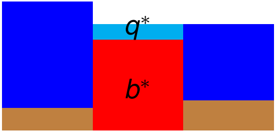
where $\phi = \frac{1}{2}gh^2 + hu^2$,
$\Psi(q_L, q_R)$ = source term arising from bathymetric variation, and
$s_\epsilon^{1,2}$, s_M = ‘speeds’, eigenvalues or their averages/min/max

Ghost State at Barrier: Redistribution



zero-width → ghost-state



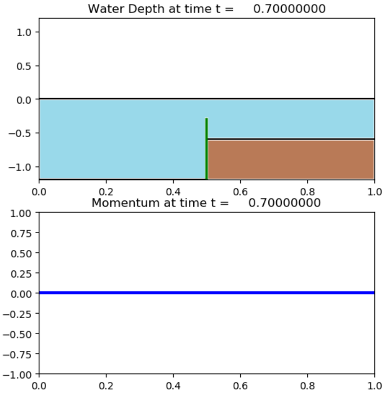
Ghost State at Barrier: Redistribution

$$b^* = \min(b_L, b_R) + \text{wall height} \tag{8}$$

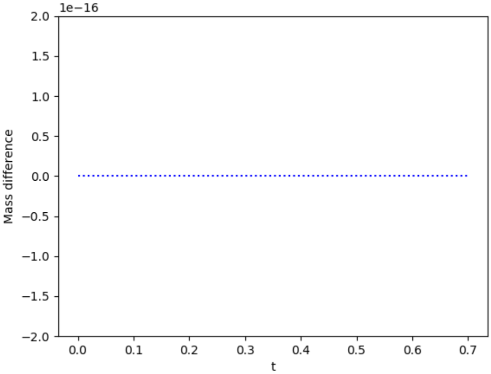
$$h^* = \min(h_L - (b^* - b_L), h_R - (b^* - b_R)) \tag{9}$$

$$(hu)^* = \min((hu)_L, (hu)_R) \tag{10}$$

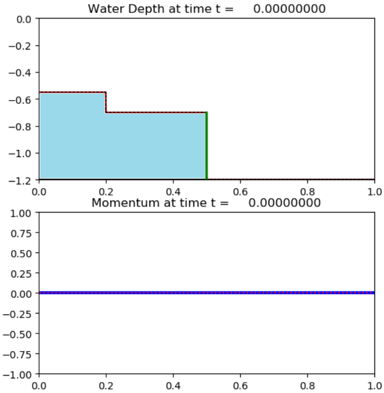
Lake at rest case



Lake at rest case



Inundation case I



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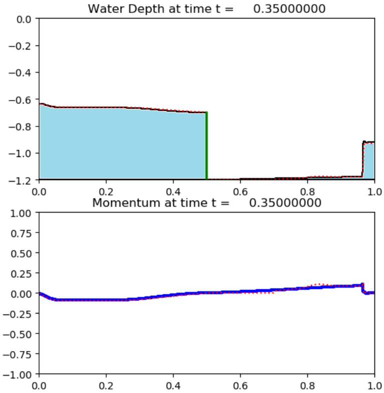
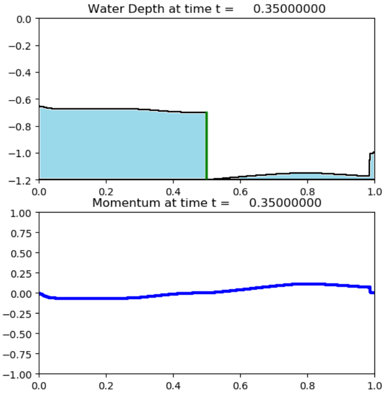
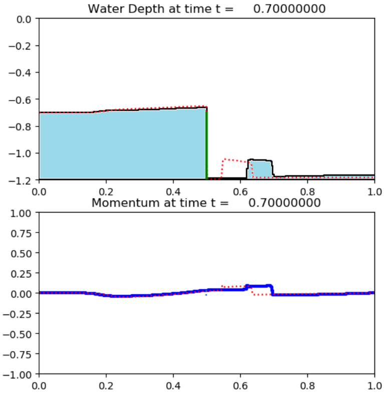


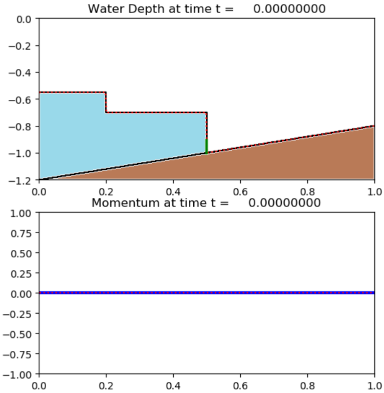
Inundation case I (comparison)



Inundation case I



Inundation case II



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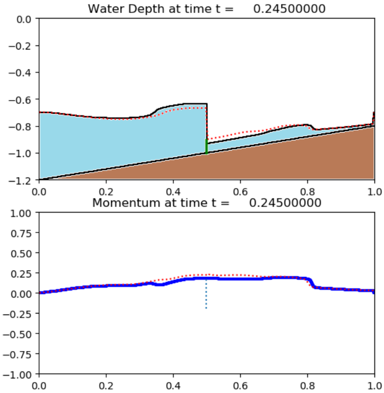

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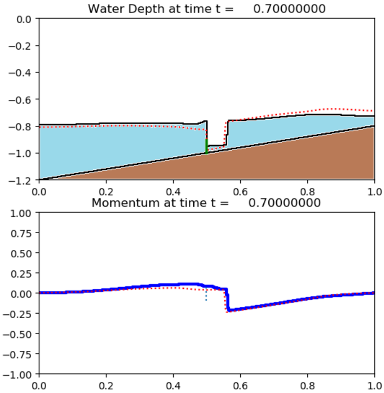
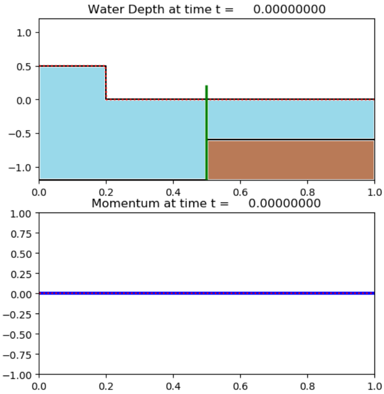


Overtopping over bathymetry jump

Outflow at right



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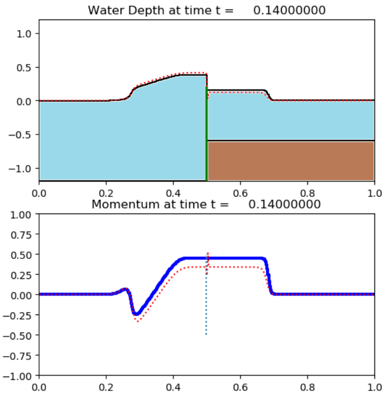

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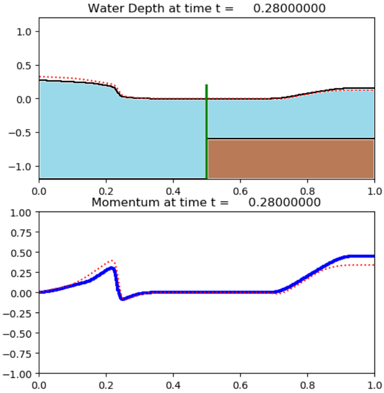

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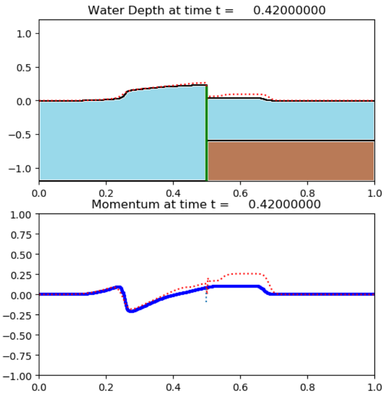

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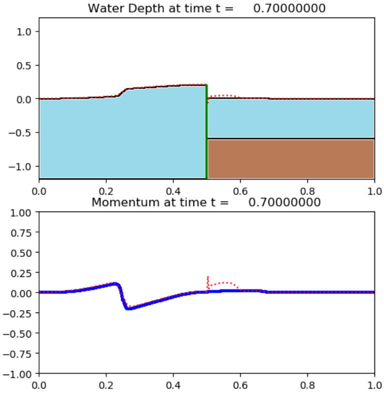
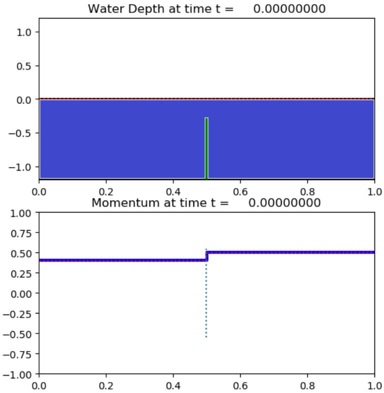


Steady state subcritical flow



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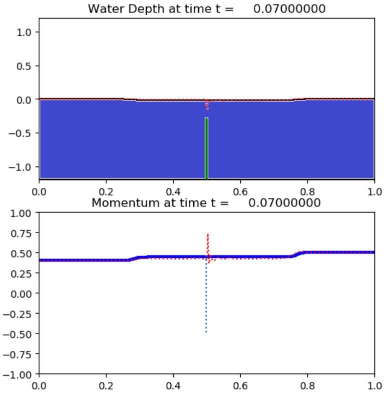

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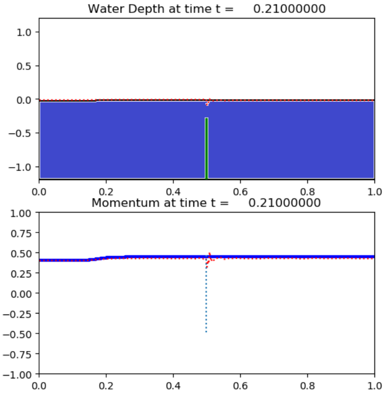

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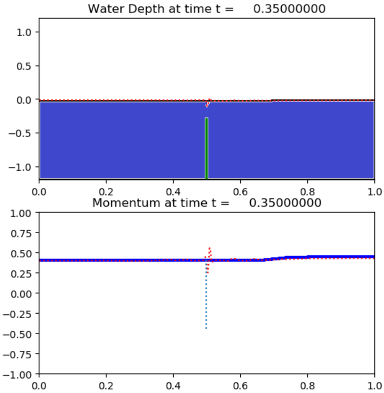

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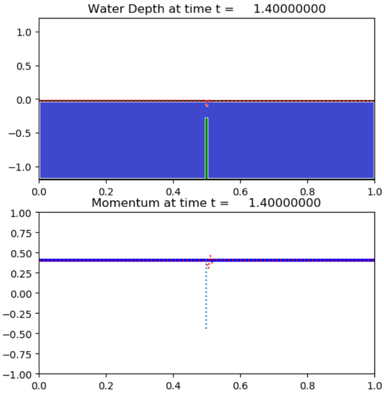

Summary

- Mass conservation observed : -7.406 E -16
- Simplified calculation
- Better on dry state conditions

- Outlook
  - Subcritical flow cannot be captured on infinitely thin wall
  - Convergence studies
  - 2D problem

For Further Reading

R. Leveque
Finite Volume Methods for Hyperbolic Problems.
Cambridge Publication, 2002.

D. George.
Augmented Riemann solvers for the SWE over variable topography with steady states and inundation
Journal of Computational Physics, 227(6), 2008.